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MATHEMATICS

On approximation processes defined by a cosine operator function

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Abstract. In this paper we introduce the Blackman- and Rogosinski-type approximation processes in an abstract Banach space setting. The historical roots of these processes go back to W. W. Rogosinski in 1926. The given new definitions use a cosine operator functions concept. We prove that in the presented setting the Blackman- and Rogosinski-type operators possess the order of approximation that coincides with results known in trigonometric approximation. Also applications for different types of approximations are given. An application for the Fourier series of symmetric functions with respect to π is emphasized.

Key words: cosine operator function, Blackman-type approximation processes, Rogosinski-type approximation processes, modulus of continuity, Fourier series of symmetric functions with respect to π.

1. INTRODUCTION

The aim of this paper is to introduce an abstract framework of certain approximation processes using a cosine operator functions concept. Historical roots of these processes go back to Rogosinski [13], who proved that the arithmetical mean of shifted Fourier partial sums converges uniformly to a given 2π-periodic continuous function $f \in C_{2\pi}$. In notations: for $f \in C_{2\pi}$ the Fourier partial sums

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

define the Rogosinski means by

$$R_n(f,x) := \frac{1}{2}\left(S_n\left(f, x + \frac{\pi}{2(n+1)}\right) + S_n\left(f, x - \frac{\pi}{2(n+1)}\right)\right). \tag{1.1}$$

Let X be an arbitrary (real or complex) Banach space and $[X]$ the Banach algebra of all bounded linear operators U of X into itself.

Let $\{P_k\}_{k=0}^{\infty} \subset [X]$ be a given sequence of mutually orthogonal projections, i.e. $P_jP_k = \delta_{jk}P_k$, (δ_{jk} being the Kronecker symbol). Moreover, let us assume that the sequence of projections is total, i.e. $P_k f = 0$ for all

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$k = 0, 1, 2, \ldots$ implies $f = 0$, and fundamental, i.e. the linear span of $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X. Then with each $f \in X$ one may associate its unique Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f$$

with the Fourier partial sums operator or Fourier projection operator

$$S_n f = \sum_{k=0}^{n} P_k f.$$

As we know from the trigonometric Fourier approximation, the strong convergence of the Fourier partial sums is not guaranteed for all $f \in X$. The improvement of that situation will be given by some matrix transformation like

$$U_n f = \sum_{k=0}^{n} \Theta_k(n) P_k f.$$

The first matrix transformation with $\Theta_k(n) = 1 - \frac{k}{n+1}$ for the trigonometric Fourier series was introduced by L. Fejér in 1904 ([5], cited in [3]). Later Rogosinski [13] introduced the arithmetical mean of shifted Fourier partial sums (1.1), which appeared to be the matrix transformation with $\Theta_k(n) = \cos \frac{k\pi}{2(n+1)}$.

In this paper we introduce in an abstract setting the Rogosinski- and Blackman-type operators and find the order of approximation via a modulus of continuity (smoothness), which is defined by a general cosine operator functions. We used a little less abstract setting in [8]. The Rogosinski- and Blackman-type operators are interesting because they are applicable in approximation by Fourier expansions of different orthogonal systems, in summation of Fourier transforms, and in approximation by generalized Shannon sampling operators. Moreover, in some particular cases, we are able to calculate precise values of their operator norms.

Definition 1. *A cosine operator function $T_h \in [X]$ $(h \geq 0)$ is defined by the properties*
(i) $T_0 = I$ *(identity operator),*
(ii) $T_{h_1} \cdot T_{h_2} = \frac{1}{2}(T_{h_1+h_2} + T_{|h_1-h_2|})$,
(iii) $\|T_h f\| \leq T\|f\|$, *the constant $T > 0$ does not depend on $h > 0$.*

Remark 1. Let $\tau_h \in [X]$, $h \in \mathbb{R}$, be a translation operator, defined by the properties
(i) $\tau_0 = I$,
(ii) $\tau_{h_1} \cdot \tau_{h_2} = \tau_{h_1+h_2}$,
(iii) $\|\tau_h f\| \leq T\|f\|, 0 < T-$ not depending on $h \in \mathbb{R}$.
Then $T_h := \frac{1}{2}(\tau_h + \tau_{-h})$, $h \geq 0$, is a cosine operator function.

The following example demonstrates why sometimes we should use the cosine operator function.

System of symmetric trigonometric functions with respect to π. Let $X = C_{2\pi}^{-}$ denote the space of symmetric functions with respect to π (shortly π-symmetric) and in addition 4π-periodic, i.e. we suppose that $f(\pi - x) = f(\pi + x)$ (or equivalently, $f(2\pi - x) = f(x)$) and $f(4\pi + x) = f(x)$ for all $x \in \mathbb{R}$. For example, for $k = 0, 1, 2, \ldots$ the functions $y = \cos kx$, $y = \sin(k + \frac{1}{2})x$ are in space $C_{2\pi}^{-}$, but $y = \sin((k+1/2)x)$ are not in $C_{2\pi}$. An interesting phenomenon of π-symmetry is that for any continuous function f on $[-\pi, \pi]$ its π-symmetric and 4π-periodic extension are always continuous on $\mathbb{R}$. This is not the case of 2π-periodic extension of any continuous function f on $[-\pi, \pi]$; to be continuous in addition the equality $f(-\pi) = f(\pi)$ should be valid. Since the system $\{\cos kx,\ \sin((k+1/2)x)\}_{k=0}^{\infty}$ is orthogonal on $[-\pi, \pi]$ under usual scalar product, we may consider the Fourier partial sums operator

$$S_n^{-}(f, x) = {\sum_{k=0}^{n}}' \left(a_k \cos kx + d_k \sin\left((k+1/2)x\right)\right), \tag{1.2}$$

where $\sum'$ here and in the following means that the coefficient a_0 is halved and

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt,\ \ d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin((k+1/2)t)dt.$$

If a function $f \in C_{2\pi}$, it is obvious that for the ordinary translation operator $\tau_h(f,x) = f(x+h)$, $h \in \mathbb{R}$, we have $\tau_h f \in C_{2\pi}$ as well. But here we may note that the ordinary translation operator $\tau_h(f,x) = f(x+h)$, $h \in \mathbb{R}$, is not good for the π-symmetric functions, since, for example, $\tau_h\left(\sin\left(\frac{1}{2}\circ\right),x\right) = \sin\frac{1}{2}(x+h) \notin C^-_{2\pi}$ for every $h \neq 2k\pi, k \in \mathbb{Z}$. For the cosine operator function

$$T_h(f,x) = \frac{1}{2}\left(f(x+h)+f(x-h)\right),\ h \geq 0$$

we state the following, quite obvious,

Lemma 1. *Let $f \in C^-_{2\pi}$. Then for every $h \geq 0$ the cosine operator function yields $T_h f \in C^-_{2\pi}$.*

Fourier-Chebyshev series. For $f \in C_{[-1,1]}$ let us consider the Fourier–Chebyshev partial sums operator

$$S^C_n(f,x) = \hat{f}_C(0) + 2\sum_{k=1}^{n} \hat{f}_C(k)T_k(x),$$

where

$$\hat{f}_C(k) := \frac{1}{\pi}\int_{-1}^{1} f(u)T_k(u)\frac{du}{\sqrt{1-u^2}}$$

is the kth Fourier–Chebyshev coefficient, and $T_k(u) = \cos(k\arccos u)$ is the kth Chebyshev polynomial of the first kind. For this case a suitable cosine operator function (see [2,4]) is

$$T^C_h(f,x) := \frac{1}{2}\left\{f(x\cos h + \sqrt{1-x^2}\sin h) + f(x\cos h - \sqrt{1-x^2}\sin h)\right\},\ 0 \leq h \leq \pi.$$

Section 2 deals with preliminary notions such as the modulus of continuity and the best approximation, but in that use an abstract setting the cosine operator function. The main definitions of the Rogosinski- and Blackman-type approximation operators are introduced. Section 3 treats the order of approximation by the Rogosinski- and Blackman-type approximation operators. Section 4 is concerned with applications to the π-symmetric trigonometric approximation operators. In Section 5 we consider a quite specific problem, namely it appears that the exact values of the operator norms of the Rogosinski- and Blackman-type π-symmetric trigonometric approximation operators can be calculated.

2. MODULUS OF CONTINUITY, BEST APPROXIMATIONS, ROGOSINSKI- AND BLACKMAN-TYPE OPERATORS

In this section we will define the Rogosinski- and Blackman-type approximation operators and the apparatus that is needed for estimating the order of approximation. The leading idea for definitions below appeared from the trigonometric approximation, see [3,12–14] and references therein.

An abstract modulus of continuity, defined by the cosine operator function, will play an important role in our paper.

Definition 2. *The modulus of continuity of order $k \in \mathbb{N}$ is defined for $\delta \geq 0$ via the cosine operator function by*

$$\omega_k(f,\delta) := \sup_{0\leq h\leq\delta} \|(T_h - I)^k f\|. \tag{2.1}$$

The next properties are adaptions of the well-known properties of the ordinary modulus of continuity (see, e.g. [3,10,14]).

Proposition 1. *The modulus of continuity $\omega_k(f,\delta)$ $(\omega(f,\delta) := \omega_1(f,\delta))$ in Definition 2 has the following properties:*
(i) $\omega(f,m\delta) \leq m(1+(m-1)T)\omega(f,\delta), m \in \mathbb{N}$;
(ii) $\omega(f,\lambda\delta) \leq ([\lambda]+1)(1+[\lambda]T)\omega(f,\delta), \lambda > 0$, $([\lambda] \leq \lambda$ *is the entire part of* $\lambda \in \mathbb{R})$;
(iii) $\omega_k(f,\delta) \leq (1+T)^{k-l}\omega_l(f,\delta), k \geq l$ *and* $k,l \in \mathbb{N}$.

Remark 2. Let $\tau_h : X \to X$, $h \in \mathbb{R}$, be a translation operator and let us define another modulus of continuity of order $k \in \mathbb{N}$ by

$$\widetilde{\omega}_k(f,\delta) := \sup_{0\leq h\leq\delta} \left\| \left(\tau_{h/2} - \tau_{-h/2}\right)^k f \right\|.$$

Then by Remark 1 $T_h := \frac{1}{2}\left(\tau_h + \tau_{-h}\right)$, $h \geq 0$, defines the modulus of continuity ω_k by (2.1). Since $T_h - I = \frac{1}{2}\left(\tau_{h/2} - \tau_{-h/2}\right)^2$, we have

$$\omega_k(f,\delta) = \frac{1}{2^k}\widetilde{\omega}_{2k}(f,\delta). \tag{2.2}$$

Another quantity we need is the best approximation. Let $A_\sigma \subset X$ be a dense family of linear subspaces with $A_{\sigma_1} \subset A_{\sigma_2}, 0 < \sigma_1 < \sigma_2$, meaning that for every $f \in X$ there exists a family $\{g_\sigma\}_{\sigma>0} \subset \bigcup_{\sigma>0} A_\sigma$ such that $\lim_{\sigma\to\infty} \|f - g_\sigma\| = 0$. Let $A_\sigma \subset X$ consist of fixed points of a linear operator $S_\sigma : A_\sigma \to A_\sigma$, i.e. for any $g \in A_\sigma$ we have $S_\sigma g = g$.

Definition 3. *The best approximation of $f \in X$ by elements of A_σ is defined by*

$$E_\sigma(f) := \inf_{g\in A_\sigma} \|f - g\|.$$

Remark 3. We may often suppose that there exists an element $g_* \in A_\sigma$ of best approximation, i.e. $E_\sigma(f) = \|f - g_*\|$.

First, let us define the Rogosinski- and Blackman-type operators only on the subspace A_σ. In the following definitions instead $S_\sigma g, (g \in A_\sigma)$, we could write just g, because by our assumption $S_\sigma g = g$. However, we prefer the given definitions, since in some cases we are able to define the operators S_σ on the whole space X, still with a set of fixed points A_σ, and in this case also the Rogosinski- and Blackman-type operators will be defined on the whole space X. To clarify the situation let us give two characteristic examples.

(1) Fourier projections are defined on the whole space X having the fixed point set as corresponding generalized polynomials.

(2) Let $X = C(\mathbb{R})$ be the space of uniformly continuous and bounded functions on $\mathbb{R}$ (for what follows, see, e.g. [6]) with a family of dense subsets $B^\infty_\sigma \subset C(\mathbb{R})$ consisting of bounded functions on $\mathbb{R}$, which are entire functions $f(z)$ $(z \in \mathbb{C})$ of exponential type σ, i.e. $|f(z)| \leq e^{\sigma|y|}\|f\|_C \quad (z = x+iy \in \mathbb{C})$. In this case the linear operator $S_\sigma : B^\infty_\sigma \to B^\infty_\sigma$ is the classical Whittaker–Kotel$'$nikov–Shannon operator, for $g \in B^\infty_\sigma, \sigma < \pi w$, defined by

$$(S^{\mathrm{sinc}}_w g)(t) := \sum_{k=-\infty}^{\infty} g\left(\frac{k}{w}\right) \mathrm{sinc}(wt - k),$$

where the kernel function $\mathrm{sinc}(t) := \frac{\sin \pi t}{\pi t}$. The fact that for $S^{\mathrm{sinc}}_w : B^\infty_\sigma \to B^\infty_\sigma$ the set of fixed points is $B^\infty_\sigma, \sigma < \pi w$, is a statement of the famous Whittaker–Kotel$'$nikov–Shannon theorem: if $g \in B^\infty_\sigma, \sigma < \pi w$, then

$$(S^{\mathrm{sinc}}_w g)(t) = g(t).$$

Definition 4. *The Rogosinski-type operators* $\widetilde{R}_{\sigma,h,a} : A_\sigma \to X$ *are defined by*

$$\widetilde{R}_{\sigma,h,a} g := aT_h(S_\sigma g) + (1-a)\,T_{3h}(S_\sigma g) \;\; (\sigma > 0, h \geq 0, a \in \mathbb{R}).$$

Remark 4. The case $a = 1$ leads to the original Rogosinski operator $R_{n,h} : C_{2\pi} \to C_{2\pi}$, which in trigonometric approximation was introduced by Rogosinski [13] and afterwards elaborated by Stechkin in [15], see also [3,14,16].

Definition 5. *The Blackman-type operators* $\widetilde{B}_{\sigma,h,a} : A_\sigma \to X$ *are defined by*

$$\widetilde{B}_{\sigma,h,a} g := aS_\sigma g + \frac{1}{2}T_h(S_\sigma g) + \left(\frac{1}{2} - a\right) T_{2h}(S_\sigma g) \;\; (\sigma > 0, h \geq 0, a \in \mathbb{R}).$$

Remark 5. In Definition 5 the Blackman operator in case $a = 1/2$ is called the Hann operator, denoted here by $\widetilde{H}_{\sigma,h}$, and in Communications Engineering it is the original Hann operator [1]. If the projector operator $S_\sigma : A_\sigma \to A_\sigma$ is translation invariant, i.e. $T_h S_\sigma = S_\sigma T_h$, then it is easy to prove that $\widetilde{R}^2_{\sigma,h} = \widetilde{H}_{\sigma,2h}$ and $\widetilde{B}_{\sigma,h,3/8} = \widetilde{H}^2_{\sigma,h}$.

The following Bounded Linear Transformation Theorem allows us to define our Rogosinski- and Blackman-type operators on the whole space X.

Theorem 1 ([7]) Sect. 8.2, 8.3)**.** *Let* $A \subset X$ *be a dense subset of a Banach space* X *and* $\widetilde{B} : A \to X$ *is a bounded linear operator with the operator norm* $\|\widetilde{B}\|$. *Then* $\widetilde{B}$ *has the unique bounded linear extension* $B : X \to X$ with $\|B\| = \|\widetilde{B}\|$. *For* $f \in X$ *the operator* $B \in [X]$ *is defined by* $Bf = \lim_{\sigma\to\infty} \widetilde{B} g_\sigma$, *where* $\{g_\sigma\}_{\sigma>0} \subset A$ *is an arbitrary family with* $f = \lim_{\sigma\to\infty} g_\sigma$.

In the following we define the Rogosinski- and Blackman-type operators on the whole X as extensions of operators given by Definitions 4 and 5, respectively, and denote them by $R_{\sigma,h,a} : X \to X$ or $B_{\sigma,h,a} : X \to X$, respectively.

3. ORDER OF APPROXIMATION BY ROGOSINSKI- AND BLACKMAN-TYPE OPERATORS

In this section we discuss the order of approximation of the Rogosinski- and Blackman-type operators by the modulus of continuity.

Theorem 2. *For every* $f \in X$, $a \in \mathbb{R}$ *for the Rogosinski-type operator* $R_{\sigma,h,a} : X \to X$ *it holds that*

$$\|R_{\sigma,h,a} f - f\| \leq \big(\|R_{\sigma,h,a}\|_{[X]} + |a|T + |1-a|T\big)\,E_\sigma(f) + |a|\omega(f,h) + |1-a|\omega(f,3h), \tag{3.1}$$

where the constant $T > 0$ *is determined by Definition 1, (iii).*

Proof. Let $g_* \in A_\sigma$ be an element of the best approximation of $f \in X$ and let us denote

$$\Theta_{h,a} f := aT_h f + (1-a)T_{3h} f. \tag{3.2}$$

Since $S_\sigma g_* = g_*$, by Definition 4 $\Theta_{h,a} g_* = \widetilde{R}_{\sigma,h,a} g_*$ and we get

$$\|\widetilde{R}_{\sigma,h,a} g_* - f\| \leq \|\Theta_{h,a} g_* - \Theta_{h,a} f\| + \|\Theta_{h,a} f - f\|. \tag{3.3}$$

For the first term in the right-hand side of (3.3) by (3.2) we obtain

$$\|\Theta_{h,a}g_* - \Theta_{h,a}f\| \leq \|\Theta_{h,a}\|_{[X]}\|g_* - f\| \leq (|a| + |1-a|)\, T E_\sigma(f). \tag{3.4}$$

For the second term in the right-hand side of (3.3) by (3.2) we write

$$\Theta_{h,a}f - f = a\,(T_h - I)\,f + (1-a)\,(T_{3h} - I)\,f,$$

thus, by Definition 2,

$$\|\Theta_{h,a}f - f\| \leq |a|\omega(f,h) + |1-a|\omega(f,3h).$$

The operator $\widetilde{R}_{\sigma,h,a} : A_\sigma \to X$ and its extension $R_{\sigma,h,a} : X \to X$ coincide on the subspace A_σ; therefore

$$\|R_{\sigma,h,a}f - f\| \leq \|R_{\sigma,h,a}f - \widetilde{R}_{\sigma,h,a}g_*\| + \|\widetilde{R}_{\sigma,h,a}g_* - f\| \leq \|R_{\sigma,h,a}\|_{[X]}E_\sigma(f) + \|\widetilde{R}_{\sigma,h,a}g_* - f\|.$$

Collecting all inequalities together we obtain the assertion. □

By property (i) of the modulus of continuity the quantities $\omega(f,h)$ and $\omega(f,3h)$ in Theorem 2 have the same order, in particular, $\omega(f,h) \leq \omega(f,3h) \leq (3+6T)\omega(f,h)$. Since $g(a) := |a| + |1-a|$ $(a \in \mathbb{R})$ has its minimum value on $[0,1]$, we specify the following

Corollary 1. *1. For* $0 \leq a < 1$ *there holds*

$$\|R_{\sigma,h,a}f - f\| \leq \left(\|R_{\sigma,h,a}\|_{[X]} + T\right) E_\sigma(f) + \omega(f,3h).$$

2. Let us denote $R_{\sigma,h} := R_{\sigma,h,1}$*. Then there holds*

$$\|R_{\sigma,h}f - f\| \leq \left(\|R_{\sigma,h}\|_{[X]} + T\right) E_\sigma(f) + \omega(f,h).$$

The importance of the parameter $a \in \mathbb{R}$ consists in the statement that because for $a = 9/8$ we get a much better order of approximation.

Theorem 3. *For* $R_{\sigma,h,9/8} : X \to X$ *we have*

$$\|R_{\sigma,h,9/8}f - f\| \leq \left(\|R_{\sigma,h,9/8}\|_{[X]} + \frac{5}{4}T\right) E_\sigma(f) + \left(1 + \frac{T}{2}\right)\omega_2(f,h).$$

Proof. Similarly to the proof of Theorem 1 we proceed

$$\|\widetilde{R}_{\sigma,h,9/8}g_* - f\| \leq \|\Theta_h g_* - \Theta_h f\| + \|\Theta_h f - f\|, \tag{3.5}$$

where this time

$$\Theta_h f := \frac{9}{8}T_h f - \frac{1}{8}T_{3h} f. \tag{3.6}$$

Since by properties of the cosine operator function (Definition 1)

$$f - \Theta_h f = \left(I + \frac{1}{2}T_h\right)(T_h - I)^2 f,$$

by Definition 2 we get

$$\|f - \Theta_h f\| \leq \left(1 + \frac{T}{2}\right)\omega_2(f,h),$$

which together with (3.4) and (3.5) gives the assertion. □

Analogous results are valid for the Blackman-type operators. We announce these theorems without proof, since in a less abstract form they are given in [8].

Theorem 4. *For every $f \in X$ and all $a \in \mathbb{R}$ for the Blackman-type operators $B_{\sigma,h,a} : X \to X$ there holds*

$$\|B_{\sigma,h,a}f - f\| \le \left(\|B_{\sigma,h,a}\|_{[X]} + |a| + \frac{T}{2} + \left|\frac{1}{2} - a\right| T \right) E_\sigma(f) + \frac{1}{4}\omega(f,h) + \frac{|1-2a|}{4}\omega(f,2h), \tag{3.7}$$

where the constant $T > 0$ is determined by Definition 1 (assumption (iii)).

Corollary 2. *Let the Hann operator $\widetilde{H}_{\sigma,h} : A_\sigma \to X$ be defined by the equation*

$$\widetilde{H}_{\sigma,h}f := \frac{1}{2}(S_\sigma f + T_h(S_\sigma f)).$$

Then for its extension $H_{\sigma,h} : X \to X$ for every $f \in X$ there holds

$$\|H_{\sigma,h}f - f\| \le \left(\frac{1+T}{2} + \|H_{\sigma,h}\|_{[X]} \right) E_\sigma(f) + \frac{1}{4}\omega(f,h).$$

Theorem 5. *For every $f \in X$ there holds*

$$\|B_{\sigma,h,5/8}f - f\| \le \left(\frac{5}{8}(1+T) + \|B_{\sigma,h,5/8}\|_{[X]} \right) E_\sigma(f) + \frac{1}{4}\omega_2(f,h). \tag{3.8}$$

As we may see by Theorems 2–5 and their corollaries, Definitions 3 and 4 deduce approximation processes, when the right-hand sides of given estimates tend to zero as $\sigma \to \infty$ and $h \to 0+$. This statement requires, among others, that the families of operator norms $\|B_{\sigma,h,a}\|_{[X]}$, $\|R_{\sigma,h,a}\|_{[X]}$ should be uniformly bounded on σ and h. From this argument follows that $\sigma > 0$ and $h \ge 0$ should be related somehow. We will consider this problem in a concrete situation, like in approximation by π-symmetric trigonometric polynomials. In our previous papers (see, e.g. [9]) we used the given framework for approximation by generalized Shannon sampling operators.

4. APPROXIMATION BY π-SYMMETRIC TRIGONOMETRIC ROGOSINSKI-TYPE OPERATORS

Let $X = C_{2\pi}^-$ be the space of π-symmetric and 4π-periodic continuous functions on $\mathbb{R}$, i.e. we suppose that $f(2\pi - x) = f(x)$ and $f(4\pi + x) = f(x)$ for all $x \in \mathbb{R}$. As it was mentioned above the π-symmetric and 4π-periodic extension of any continuous function f on $[-\pi,\pi]$ has the remarkable property that it is always continuous on $\mathbb{R}$. Under the norm $\|f\|_C := \max\{|f(x)| : -\pi \le x \le \pi\}$ the space $C_{2\pi}^-$ is a Banach space with a sequence of dense subspaces $\{A_n\}, n = 0,1,2,...$, consisting of (semi-integer) trigonometric polynomials of the form

$$t_n(x) = \sum_{k=0}^{n} \left(C_k \cos kx + D_k \sin\left((k+1/2)x\right)\right), \quad C_k, D_k \in \mathbb{R}.$$

As in the classical case, the density of $\{A_n\}, n = 0,1,2,...$, in $C_{2\pi}^-$ can be proved using the Fejér means [11]

$$\sigma_n(f,x) = {\sum_{k=0}^{n}}' \left(\left(1 - \frac{k}{n+1}\right) a_k \cos kx + \left(1 - \frac{k+1/2}{n+1}\right) d_k \sin\left((k+1/2)x\right) \right).$$

By orthogonality it is also clear that A_n is a set of fixed points of the Fourier partial sums operator $S_n^- : C_{2\pi}^- \to A_n$.

Concerning the cosine operator function $T_h : C_{2\pi}^- \to C_{2\pi}^-$,

$$T_h(f,x) = \frac{1}{2}\left(f(x+h) + f(x-h)\right),\ h \ge 0,$$

we have $\|T_h f\|_C \le \|f\|_C$; therefore in Definition 1 $T = 1$. Moreover,

$$T_h\left(S_n^- f, x\right) = {\sum_{k=0}^{n}}' \left(a_k \cos kh \cos kx + d_k \cos\left((k+1/2)h\right) \sin\left((k+1/2)x\right)\right). \tag{4.1}$$

Since all assumptions of our abstract framework have been discussed, we may, using Definition 4 and Eq. (4.1), state for the Rogosinski-type operators the following.

Proposition 2. *Let $f \in C_{2\pi}^{-}$, then*

$$R_{n,h,a}(f,x) = \sum_{k=0}^{n}{}' \left(\psi_a \left(\frac{2}{\pi} kh \right) a_k \cos kx + \psi_a \left(\frac{2}{\pi} (k+1/2)\, h \right) d_k \sin((k+1/2)x) \right),$$

where

$$\psi_a(t) := a \cos \frac{\pi t}{2} + (1-a) \cos \frac{3\pi t}{2}.$$

Theorem 2 yields the following statement.

Theorem 6. *For every $f \in C_{2\pi}^{-}$, $a \in \mathbb{R}$, for the Rogosinski-type operators $R_{n,h,a} : C_{2\pi}^{-} \to C_{2\pi}^{-}$ there holds*

$$\|R_{n,h,a} f - f\|_C \leq \left(\|R_{n,h,a}\|_{[C_{2\pi}^{-}]} + |a| + |1-a| \right) E_n(f) + |a| \omega(f,h) + |1-a| \omega(f,3h). \tag{4.2}$$

In an analogous way many other theorems from the abstract setting can be reformulated for the π-symmetric approximation.

Now we are going to the crucial problem how we should relate the parameters $n \in \mathbb{N}$ and $h > 0$ to guarantee that the right-hand side of (4.2) will tend to zero and at the same time the operator norms $\|R_{n,h,a}\|_{[C_{2\pi}^{-}]}$ will be uniformly bounded. The classical trigonometric approximation suggests $h = O(1/n)$, but of course for nice results there is too much freedom here. We will discuss this problem in the next section.

5. EXACT VALUES OF NORMS OF π-SYMMETRIC ROGOSINSKI-TYPE OPERATORS

Let us consider the 2π-periodic trigonometric polynomial operator (or summability operator)

$$U_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} \varphi\left(\frac{k}{n+1} \right) (a_k \cos kx + b_k \sin kx),$$

where $\varphi \in C_{[0,1]}, \varphi(0) = 1, \varphi(1) = 0$. It is known [16] that U_n transforms the space $C_{2\pi}$ into $C_{2\pi}$, and the norm $\|U_n\|_{[C_{2\pi}]}$ which satisfies

$$\sup_n \|U_n\|_{[C_{2\pi}]} = \int_{-\infty}^{\infty} |s(u)| du, \tag{5.1}$$

where the kernel function $s \in L^1(\mathbb{R})$ is given by

$$s(u) = \int_0^1 \varphi(t) \cos(\pi t u) dt. \tag{5.2}$$

Moreover, it is also known (see, e.g. [10]) that in some cases, and especially for Rogosinski- and Blackman-type operators, the polynomial operator $U_n : C_{2\pi} \to C_{2\pi}$ can be rewritten as the singular integral operator of the form

$$U_n(f,x) = \int_{-\infty}^{\infty} s(u) f\left(x - \frac{\pi u}{2(n+1)} \right) du. \tag{5.3}$$

We shall use similar ideas for the π-symmetric approximation. We begin with a lemma, with is quite obvious from the geometrical point of view.

Lemma 2. *If f is π-symmetric and integrable on $[-\pi, \pi]$, then*

$$\int_{-\pi}^{\pi} f(t) dt = \frac{1}{2} \int_{-2\pi}^{2\pi} f(t) dt.$$

For $f \in C_{2\pi}^{-}$ a suitable approximation operator $U_n \in [C_{2\pi}^{-}]$ will be defined by

$$U_n(f,x) = \sum_{k=0}^{n}{}' \left(\lambda_k(n) a_k \cos kx + \mu_k(n) d_k \sin\left((k+1/2)x\right)\right), \tag{5.4}$$

which, using Lemma 2 and substituting the Fourier coefficients a_k, d_k, can be rewritten as follows.

Proposition 3. *For $f \in C_{2\pi}^{-}$ we have*

$$U_n(f,x) = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x-t) P_n(t)\,dt,$$

where

$$P_n(t) := \sum_{k=0}^{n}{}' \left(\lambda_k(n) \cos kt + \mu_k(n) \cos\left((k+1/2)t\right)\right).$$

In particular, for the Fourier partial sums operator we have

$$S_n^{-}(f,x) = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(x-t) P_n^S(t)\,dt,$$

where

$$P_n^S(t) := \sum_{k=0}^{n}{}' \left(\cos kt + \cos\left((k+1/2)t\right)\right) = \frac{\sin\left(n+\frac{3}{4}\right)t}{2\sin\left(\frac{t}{4}\right)}.$$

Now we represent $S_n^{-} f$ in the integral form over $\mathbb{R}$ (cf., e.g. [10]). It will be achieved by using the representation (see, e.g. [17])

$$\frac{1}{\sin z} = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - \pi^2 k^2}, \quad \frac{z}{\pi} \notin \mathbb{Z}.$$

After some calculations, using the given representation for $\frac{1}{\sin(t/4)}$, we may formulate

Proposition 4. *If $f \in L^1(\mathbb{R})$, then*

$$S_n(f,x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin\left(\left(n+\frac{3}{4}\right)t\right)}{t} f(x-t)\,dt. \tag{5.5}$$

Define $R_{n,h} f := T_h(S_n f)$, then by Definition 4 of the Rogosinski-type operators we have

$$R_{n,h,a} f = a R_{n,h} f + (1-a) R_{n,3h} f. \tag{5.6}$$

By the definition of cosine operator function T_h and (5.5) we get

$$R_{n,h}(f,x) = T_h\left(S_n f, x\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin\left(\left(n+\frac{3}{4}\right)t\right)}{t} \left(f(x+h-t) + f(x-h-t)\right) dt.$$

After changing the variable of integration

$$R_{n,h}(f,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\left(n+\frac{3}{4}\right)(t+h)\right)}{t+h} + \frac{\sin\left(\left(n+\frac{3}{4}\right)(t-h)\right)}{t-h} \right) f(x-t)\,dt. \tag{5.7}$$

To proceed further we need to specify the parameter $h > 0$, and a natural choice could be

$$h = \frac{\pi}{2(n+3/4)},$$

because then

$$\sin\left((n+3/4)\left(t\pm\frac{\pi}{2(n+3/4)}\right)\right)=\mp\sin\left((n+3/4)\left(t\pm\frac{3\pi}{2(n+3/4)}\right)\right)=\pm\cos\left((n+3/4)t\right),$$

and for this specific value $h=\frac{\pi}{2(n+3/4)}$ by (5.7) we obtain

$$R_{n,h}(f,x)=\frac{h}{\pi}\int_{-\infty}^{\infty}\frac{\cos\left((h+3/4)t\right)}{h^2-t^2}f(x-t)dt,$$

$$R_{n,3h}(f,x)=\frac{3h}{\pi}\int_{-\infty}^{\infty}\frac{\cos\left((h+3/4)t\right)}{t^2-9h^2}f(x-t)dt.$$

It is important that both integrals do exist for any bounded function f on $\mathbb{R}$. Therefore, for (5.6) we obtain

$$R_{n,h,a}(f,x)=\frac{h}{\pi}\int_{-\infty}^{\infty}\frac{(12a-3)h^2+(3-4a)t^2}{(h^2-t^2)(9h^2-t^2)}\cos\left((n+3/4)t\right)f(x-t)\,dt. \tag{5.8}$$

Changing the variable of integration by $t=hs$ and putting $h=\frac{\pi}{2(n+3/4)}$ we get

$$R_{n,a}(f,x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(12a-3)+(3-4a)s^2}{(1-s^2)(9-s^2)}\cos\frac{\pi s}{2}\,f\left(x-\frac{\pi s}{2(n+3/4)}\right)ds. \tag{5.9}$$

As in 2π-periodic case, it can be proved that the Rogosinski-type operator above $R_{n,a}\in[C_{2\pi}^{-}]$ has the operator norm

$$\sup_n\|R_{n,a}\|_{[C_{2\pi}^{-}]}=\int_{-\infty}^{\infty}|s_a(u)|du, \tag{5.10}$$

where the kernel function $s_a\in L^1(\mathbb{R})$ is given by

$$s_a(t)=\frac{(12a-3)+(3-4a)t^2}{\pi(1-t^2)(9-t^2)}\cos\frac{\pi t}{2}. \tag{5.11}$$

Coincidentally, the kernel function (5.11) is exactly the same as for 2π-periodic Rogosinski-type operators, which we considered in our previous paper [8]. Therefore, we may reformulate some theorems of the 2π-periodic Rogosinski-type operators for the π-symmetric case. Before going into details we have to introduce the modified integral sine as

$$\mathrm{Sci}(x):=\int_0^x\mathrm{sinc}(t)dt. \tag{5.12}$$

A selection of results follows (cf. [8]).

Theorem 7. *The Rogosinski-type operators $R_{n,a}\in[C_{2\pi}^{-}]$ defined by the kernel function (5.11) have the following operator norms:*
1. If $0\le a\le\frac{1}{4}$, then the polynomial $p(t)=12a-3+(12-16a)t^2$ has a positive zero t_a and

$$\sup_n\|R_{n,a}\|_{[C_{2\pi}^{-}]}=-2\int_0^{t_a}s_a(t)dt+2\int_{t_a}^{1/2}s_a(t)dt+(3-2a)Sci(1)+(5a-3)Sci(2)+(a-1)(Sci(4)-3Sci(3)).$$

2. If $\frac{1}{4}\le a\le\frac{3}{4}$, then $(p(t)\ge 0)$ and

$$\sup_n\|R_{n,a}\|_{[C_{2\pi}^{-}]}=2\int_0^{1/2}s_a(t)dt+(1-2a)Sci(1)+(5a-3)Sci(2)+(2a-2)Sci(3).$$

Corollary 3. *The operator norms of the Rogosinski-type operators have numerical values:*
1. If $a=\frac{3}{4}$, then $\sup_n\|R_{n,\frac{3}{4}}\|_{[C_{2\pi}^{-}]}=1.88903...$
2. If $a=\frac{1}{2}$, then $\sup_n\|R_{n,\frac{1}{2}}\|_{[C_{2\pi}^{-}]}=1.39741...$

6. CONCLUSION

We introduced the Blackman- and Rogosinski-type approximation operators using the cosine operator function. This abstract setting is useful because now we were able to consider different approximation problems from a unique point of view. Another feature of this paper was that in π-symmetric trigonometric case we computed exact values of some operator norms of the defined Rogosinski-type approximation operators.

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Koosinusoperaatori fuktsioonidega defineeritud lähendusmeetoditest

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On defineeritud abstraktses Banachi ruumis Blackmani ja Rogosinski tüüpi operaatorid, kasutades koosinus-operaatori mõistet. Uus lähenemisviis võimaldab ühtsest seisukohast tõestada lähenduskiiruste hinnanguid ja rakendada saadud tulemusi Shannoni valimoperaatoritele, trigonomeetrilistele Fourier' ridadele või ka Fourier'-Tšebõšovi ridadele. Trigonomeetrilisel π-sümmeetrilisel juhul defineeritud Rogosinski tüüpi mõnede lähendusoperaatorite jaoks on leitud täpsed normi väärtused.