Proceedings of the Estonian Academy of Sciences,
2017, **66**, 2, 137–150
https://doi.org/10.3176/proc.2017.2.04
Available online at www.eap.ee/proceedings

MATHEMATICS

# On endomorphisms of groups of orders 37–47

Alar Leibak* and Peeter Puusemp

Department of Cybernetics, School of Science, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia

Received 7 June 2016, revised 20 December 2016, accepted 8 February 2017, available online 6 April 2017



**Abstract.** It is proved that the finite groups of orders 37–47 are determined by their endomorphism monoids in the class of all groups.

**Key words:** group, monoid, endomorphism monoid.

## 1. INTRODUCTION

Let $G$ be a group. If for each group $H$ such that the monoids $\mathrm{End}(G)$ and $\mathrm{End}(H)$ are isomorphic implies an isomorphism between $G$ and $H$, we say that the group $G$ is determined by its endomorphism monoid in the class of all groups. Examples of such groups are: finite Abelian groups ([12], Theorem 4.2), generalized quaternion groups ([13], Corollary 1), torsion-free divisible Abelian groups ([16], Theorem 1), etc.

The endomorphisms of groups have gained much attention in the past few years due to their applications in generalized linear finite dynamical systems (in this case, the finite vector space and a polynomial map $f$ are replaced with a group and its endomorphism, respectively) [3]. Also, Grigorchuk and Mamaghani [7] used iterations of an endomorphism of a group for constructing the groups with prescribed properties such as to have intermediate growth or to be amenable. Different authors have studied so-called $E$-groups (recall that a group $G$ is called an $E$-group if its each element commutes with all of images under endomorphisms of $G$).

The question of which groups are determined by their endomorphism monoids in the class of all groups, and how to find all non-isomorphic groups with isomorphic endomorphism monoids, is one of the main questions in the group theory. For example, in 2014 the following question was asked in the internet forum math.stackexchange: “Can finite non-isomorphic groups of the same order have isomorphic endomorphism monoids?” [4].

In a number of our papers we have made efforts to describe some classes of finite groups which are determined by their endomorphism monoids in the class of all groups.

* Corresponding author, alar.leibak@ttu.ee

Computer algebra software GAP [24] provides access to small groups[1] such as the presentation of a group and structure description of a group. The software GAP is becoming a useful tool for mathematicians working in the group theory. For example, Chu et al. [6] used the GAP for solving Noether's problem for groups of order 243. Combining the software GAP with the results obtained in [12–14,18,19] it is possible to obtain the list of groups of given order $n$ that are determined by their endomorphism monoids. However, the groups of order $n$ not listed there have to be studied using techniques different from the ones used in [12–14,18,19] or ad-hoc techniques. Recall that the finite direct product of groups is determined by its endomorphism monoid if all these direct factors are determined by their endomorphism monoids [12, Theorem 1.13]. Therefore, if a group $G$ is the direct product of groups $G_1, \ldots, G_n$, and each of them are determined by their endomorphism monoids, then so is $G$. But if a direct factor, say $G_i$, is not determined by its endomorphism monoid, i.e. $\mathrm{End}(G_i) \cong \mathrm{End}(H)$ for some group $H \ncong G_i$, then the direct products $G_1 \times \cdots \times G_n$ and $G_1 \times \cdots \times G_{i-1} \times H \times G_{i+1} \times \cdots \times G_n$ can have isomorphic endomorphism monoids. For example, it was shown in [20] that the alternating group $A_4$ and the binary tetrahedral group $\mathscr{B} \cong \mathrm{SL}_2(\mathrm{GF}(3))$ have isomorphic endomorphism monoids. Studying the determinability of groups of order 36, we proved that the direct products $C_3 \times A_4$ and $C_3 \times \mathscr{B}$ have isomorphic endomorphism monoids too [9, Theorem 6.1]. On the other hand, if a finite group $G$ is a direct product of all pairwise non-isomorphic groups with isomorphic endomorphism monoids, then the group $G$ is determined by $\mathrm{End}(G)$ [15]. Therefore, the direct product $A_4 \times \mathscr{B}$ is determined by its endomorphism monoid, but both direct factors are not. In what follows we discuss a simple[2] (and naive) method for testing whether a given direct product of finite groups is determined by its endomorphism monoid or not: either all direct factors are determined by their endomorphism monoids, respectively, or the set of direct factors includes a complete set of pairwise non-isomorphic groups with isomorphic endomorphism monoids. This motivates us to study which small groups are determined by their endomorphism monoids. Furthermore, since a similar technique can be applied to several groups of different order, we are also motivated to use a computer[3] and development of necessary algorithms. The results of this paper can be used in computer programs about finite groups and their applications.

We know a complete answer to this problem for finite groups of order less than 37. Among the groups of order less than 37 there are only five groups that are not determined by their endomorphism monoids in the class of all groups:

- the alternating group $A_4$ (also called the tetrahedral group);
- the binary tetrahedral group $\mathscr{B} = \langle a, b \mid b^3 = 1,\ aba = bab\rangle$ ($|\mathscr{B}| = 24$);
- the direct product $C_3 \times A_4$ of the cyclic group of order 3 and the alternating group $A_4$;
- $\mathscr{C}_1 = \langle a, b, c \mid c^9 = b^2 = a^2 = 1,\ ab = ba,\ c^{-1}ac = b, c^{-1}bc = ab\rangle = (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle \cong (C_2 \times C_2) \rtimes C_9$;
- $\mathscr{C}_2 = \langle a, b, c \mid c^4 = a^3 = b^3 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = a^{-1}\rangle = (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle \cong (C_3 \times C_3) \rtimes C_4$.

The alternating group $A_4$ (also called the tetrahedral group) and the binary tetrahedral group $B = \langle a, b \mid b^3 = 1,\ aba = bab\rangle$ are the only groups of order less than 36 that are not determined by their endomorphism monoids in the class of all groups [9,20–23] (for some groups of order 32 the proofs are under publishing). In [9], it is proved that among groups of order 36 only the groups $C_3 \times A_4$, $\mathscr{C}_1$, and $\mathscr{C}_2$ are not determined by their endomorphism monoids in the class of all groups. Namely, the following results were proved in [9]:

- The endomorphism monoid of a group $G$ is isomorphic to the endomorphism monoid of $C_3 \times A_4$ if and only if $G = C_3 \times A_4$ or $G = C_3 \times \mathscr{B}$, where $\mathscr{B}$ is the binary tetrahedral group.
- The endomorphism monoid of a group $G$ is isomorphic to the endomorphism monoid of $\mathscr{C}_1$ if and only if $G = \mathscr{C}_1$ or $G$ is isomorphic to the following group of order 72:

$$\langle a, b, c \mid c^9 = a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1},\ c^{-1}bc = a, c^{-1}ac = ab\rangle$$

[1] Access is provided by the Small Groups Library http://www.icm.tu-bs.de/ag_algebra/software/small/small.html.

[2] The method explained here can be easily programmed and is based on the so-called table lookup.

[3] The most famous problem in group theory the solving of which involves the use of computers is "the general problem for finite groups" initiated by A. Cayley [5] in 1878 ([2,8,10,11]).

(this group is isomorphic to a semidirect product $Q \rightthreetimes C_9$ of the quaternion group $Q$ and the cyclic group $C_9$ of order 9).

- The endomorphism monoid of a group $G$ is isomorphic to the endomorphism monoid of $\mathscr{C}_2$ if and only if $G = \mathscr{C}_2$ or $G$ is isomorphic to the following group of order 108:

$$G = \langle a, b, c, d \mid c^4 = a^3 = b^3 = d^3 = 1,\ ab = bad,\ c^{-1}ac = b,$$

$$c^{-1}bc = a^{-1},\ cd = dc,\ ad = da,\ bd = db \rangle.$$

In this paper, we give the solution to the problem for the groups of orders 37–47. We prove the following theorem:

**Theorem 1.1** (**Main theorem**). *The finite groups of orders 37–47 are determined by their endomorphism monoids in the class of all groups.*

The proof of the theorem follows from Theorems 3.1, 4.1, 5.2, and 6.2.

We shall use the following notations:

$G$ – a group;
$Z(G)$ – the centre of a group $G$;
$N_M(g)$ – the normalizer of $g$ in $M$ ($M \subset G$);
$\mathrm{End}(G)$ – the endomorphism monoid of $G$;
$C_k$ – the cyclic group of order $k$;
$A_4$ – the alternating group of order 12 (the tetrahedral group);
$D_n = \langle a, b \mid b^2 = a^k = 1,\ b^{-1}ab = a^{-1} \rangle$ – the dihedral group of order $n = 2k$;
$\mathbb{Z}_k$ – the residue class ring $\mathbb{Z}/k\mathbb{Z}$;
$\langle K, \ldots, g, \ldots \rangle$ – the subgroup generated by subsets $K, \ldots$ and elements $g, \ldots$;
$\widehat{g}$ – the inner automorphism of $G$, generated by an element $g \in G$;
$I(G)$ – the set of all idempotents of $\mathrm{End}(G)$;
$K(x) = \{z \in \mathrm{End}(G) \mid zx = xz = z\}$;
$J(x) = \{z \in \mathrm{End}(G) \mid zx = xz = 0\}$;
$V(x) = \{z \in \mathrm{Aut}(G) \mid zx = x\}$;
$H(x) = \{z \in \mathrm{End}(G) \mid xz = z,\ zx = 0\}$;
$P(x) = \{z \in \mathrm{End}(G) \mid xz = zx = x\}$;
$[x] = \{z \in I(G) \mid xz = z,\ zx = x\},\ x \in I(G)$;
$K_0(x)$ – the set of all nilpotent elements of $K(x)$;
$G = A \rightthreetimes B$ – $G$ is a semidirect product of an invariant subgroup $A$ and a subgroup $B$.

The sets $K(x), V(x), H(x), P(x)$, and $J(x)$ are submonoids of $\mathrm{End}(G)$, furthermore, $V(x)$ is a subgroup of $\mathrm{Aut}(G)$. We shall write the mapping right from the element on which it acts.

## 2. PRELIMINARIES

For the convenience of the reader, let us recall some known facts that will be used in the proofs of our main results.

**Lemma 2.1.** *If* $x \in I(G)$, *then* $G = \mathrm{Ker}\, x \rightthreetimes \mathrm{Im}\, x$ *and* $\mathrm{Im}\, x = \{\, g \in G \mid gx = g \,\}$.

**Lemma 2.2.** *If* $x \in I(G)$, *then* $[x] = \{y \in I(G) \mid \mathrm{Ker}\, x = \mathrm{Ker}\, y\}$.

**Lemma 2.3.** *If $x \in I(G)$, then*

$$K(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}\, x)y \subset \mathrm{Im}\, x,\ (\mathrm{Ker}\, x)y = \langle 1 \rangle \,\}$$

*and $K(x)$ is a submonoid with the identity element $x$ of $\mathrm{End}(G)$ which is canonically isomorphic to $\mathrm{End}(\mathrm{Im}\, x)$. Under this isomorphism element $y$ of $K(x)$ corresponds to its restriction onto the subgroup $\mathrm{Im}\, x$ of $G$.*

**Lemma 2.4.** *If $x \in I(G)$, then*

$$H(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}\, x)y \subset \mathrm{Ker}\, x,\ (\mathrm{Ker}\, x)y = \langle 1 \rangle \,\}.$$

**Lemma 2.5.** *If $x \in I(G)$, then*

$$J(x) = \{z \in \mathrm{End}(G) \mid (\mathrm{Im}\, x)z = \langle 1 \rangle,\ (\mathrm{Ker}\, x)z \subset \mathrm{Ker}\, x\}.$$

**Lemma 2.6.** *If $z \in \mathrm{End}(G)$ and $\mathrm{Im}\, z$ is Abelian, then $\widehat{g} \in V(z)$ for each $g \in G$.*

**Lemma 2.7.** *If $x \in I(G)$, then*

$$P(x) = \{\, y \in \mathrm{End}(G) \mid y|\mathrm{Im}\, x = 1_{\mathrm{Im}\, x},\ (\mathrm{Ker}\, x)y \subset \mathrm{Ker}\, x \,\}.$$

**Lemma 2.8.** *If $y \in \mathrm{End}(G)$ and $g \in \mathrm{Ker}\, y$, then $\widehat{g} \in V(y)$.*

We omit the proofs of these lemmas, because these are straightforward corollaries from the definitions.

**Lemma 2.9** ([12], Theorem 1.13)**.** *If $G$ and $H$ are groups such that their endomorphism monoids are isomorphic and $G$ splits into a direct product $G = G_1 \times G_2$ of its subgroups $G_1$ and $G_2$, then $H$ splits into a direct product $H = H_1 \times H_2$ of its subgroups $H_1$ and $H_2$ such that $\mathrm{End}(G_1) \cong \mathrm{End}(H_1)$ and $\mathrm{End}(G_2) \cong \mathrm{End}(H_2)$.*

From here follows Lemma 2.10.

**Lemma 2.10.** *If groups $G_1$ and $G_2$ are determined by their endomorphism monoids in the class of all groups, then so is their direct product $G_1 \times G_2$.*

**Lemma 2.11** ([12], Theorem 4.2)**.** *Every finite Abelian group is determined by its endomorphism monoid in the class of all groups.*

**Lemma 2.12** ([14], Theorem)**.** *Each finite symmetric group is determined by its endomorphism monoid in the class of all groups.*

**Lemma 2.13** ([19], Section 5)**.** *The dihedral group $D_n$ is determined by its endomorphism monoid in the class of all groups.*

**Lemma 2.14** ([13], Corollary 1)**.** *The quaternion group $Q$ is determined by its endomorphism monoid in the class of all groups.*

**Lemma 2.15** ([18], Theorem)**.** *A semidirect product $G = C_{p^n} \leftthreetimes C_m$, where $p$ is a prime, and $n$ and $m$ are some positive integers, is determined by its endomorphism monoid in the class of all groups.*

**Lemma 2.16.** *If*

$$D_8 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ ab = ba,\ c^{-1}ac = b\rangle \tag{2.1}$$

*and $x \in I(D_8)$ such that* $\operatorname{Im} x = \langle c\rangle$, $\operatorname{Ker} x = \langle a\rangle \times \langle b\rangle$, *then* $1^0$ $K(x) \cong \operatorname{End}(C_2)$ *and* $2^0$ $|\{u \in \operatorname{End}(D_4) \mid xu = u,\ ux = 0\}| = 4$. *Conversely, if $x \in I(D_8)$ satisfies $1^0$ and $2^0$, then there exist $a, b, c \in D_8$ such that*

$$D_8 = \operatorname{Ker} x \leftthreetimes \operatorname{Im} x,\ \operatorname{Im} x = \langle c\rangle \cong C_2,\ \operatorname{Ker} x = \langle a\rangle \times \langle b\rangle \cong C_2 \times C_2$$

*and (2.1) holds.*

Lemma 2.16 is obtained by easy calculations in the group $D_8$.

**Lemma 2.17** ([17], Theorems 2.1 and 3.1)**.** *Assume that a group $G$ decomposes into a semidirect product*

$$G = H \leftthreetimes ((G_1 \times \ldots \times G_n) \leftthreetimes K),\ n \geq 2, \tag{2.2}$$

*where*

$$\langle G_i, K\rangle = G_i \leftthreetimes K\ \ (i = 1, 2, \ldots, n). \tag{2.3}$$

*Denote by $x$ and $x_i$ the projections of $G$ onto $K$ and $G_i \leftthreetimes K$ ($i = 1, 2, \ldots, n$), i.e.,*

$$\operatorname{Im} x_i = G_i \leftthreetimes K,\ \ \operatorname{Ker} x_i = H \leftthreetimes \prod_{j=1,\, j\neq i}^{n} G_j, \tag{2.4}$$

$$\operatorname{Im} x = K,\ \ \operatorname{Ker} x = H \leftthreetimes (G_1 \times \ldots \times G_n), \tag{2.5}$$

$$G_i = \operatorname{Ker} x \cap \operatorname{Im} x_i,\ \ H = \cap_{j=1}^{n} \operatorname{Ker} x_j. \tag{2.6}$$

*Then*

$$x_i x_j = x_j x_i = x;\ i, j = 0, 1, \ldots, n,\ i \neq j, \tag{2.7}$$

*and for each $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, there exists $z_{ij} = z_{ji} \in I(G)$ which satisfies the following properties:*
$1^0$ *$x_i, x_j \in K(z_{ij})$,*
$2^0$ *there exists a unique pair $V_i, V_j$ of subgroups of $K(z_{ij})^*$ with properties*
*(i)* $V_i \subset C(x_i),\ V_j \subset C(x_j)$,
*(ii)* $V_i x_i = V_{K(x_i)^*}(x),\ V_j x_j = V_{K(x_j)^*}(x)$,
*(iii)* *$x_i v x_i = x_i$ for each $v \in V_j$,*
*(iv)* *$x_j u x_j = x_j$ for each $u \in V_i$.*

*Conversely, suppose that there exist idempotents $x, x_1, \ldots, x_n$ of* $\operatorname{End}(G)$ *such that (2.7) holds and for each $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, there exists $z_{ij} = z_{ji} \in I(G)$ which satisfies properties $1^0$ and $2^0$. Then the group $G$ decomposes into the semidirect product (2.2), where equalities (2.3)–(2.6) are true. Moreover, the set $B = \{y \in I(G) \mid x_1, \ldots, x_n \in K(y)\}$ is non-empty and there exists a unique $z \in B$ such that $zy = yz = z$ for each $y \in B$. The endomorphism $z$ is the projection of $G$ onto its subgroup $(G_1 \times \ldots \times G_n) \leftthreetimes K$ and* $\operatorname{Ker} z = H$.

Denote by $\mathscr{C}(x; x_1, \ldots, x_n)$ the set of the conditions for $x; x_1, \ldots, x_n$ given in the second part of Lemma 2.17 (i.e., equalities (2.7) and $1^0$, $2^0$). Suppose that the condition $\mathscr{C}(x; x_1, \ldots, x_n)$ is satisfied and denote by $\pi_{\mathscr{C}}$ the projection of $G$ onto its subgroup $(G_1 \times \ldots \times G_n) \leftthreetimes K$. The endomorphism $\pi_{\mathscr{C}}$ is a unique $z \in B$ such that $zy = yz = z$ for each $y \in B$. Denote by $\mathscr{C}_0(x; x_1, \ldots, x_n)$ the condition $\mathscr{C}(x; x_1, \ldots, x_n)$ with $\pi_{\mathscr{C}} = 1_G$ (i.e., $H = \langle 1\rangle$).

## 3. GROUPS OF ORDERS 37–39 AND 41–47

The group theoretical computer algebra system GAP provides access to descriptions of small order groups [12]. Following [12], the groups of orders 37–39 and 41–47 are:
$G_1 \cong C_{37},\ |G_1| = 37;$
$G_2 \cong C_{38},\ G_3 \cong D_{38},\ |G_2| = |G_3| = 38;$
$G_4 \cong C_{39},\ |G_4| = 39;$
$G_5 = \langle a, b \mid a^3 = b^{13} = 1,\ a^{-1}ba = b^3\rangle \cong C_{13} \rtimes C_3,\ |G_5| = 39;$
$G_6 \cong C_{41},\ |G_6| = 41;$
$G_7 \cong C_{42},\ |G_7| = 42;$
$G_8 = \langle a, b, c \mid a^2 = b^3 = c^7 = (ac)^2 = b^{-1}aba = c^{-1}b^{-1}cbc^{-1} = 1\rangle,\ |G_8| = 42;$
$G_9 = \langle a, b, c \mid a^2 = b^3 = c^7 = 1,\ ab = ba,\ ac = ca,\ b^{-1}cb = c^2\rangle$
$\quad = \langle a\rangle \times (\langle c\rangle \rtimes \langle b\rangle) \cong C_2 \times (C_7 \rtimes C_3),\ |G_9| = 42;$
$G_{10} = C_7 \times S_3,\ G_{11} = C_3 \times D_{14},\ G_{12} = D_{42},\ |G_{10}| = |G_{11}| = |G_{12}| = 42;$
$G_{13} \cong C_{43},\ |G_{13}| = 43;$
$G_{14} \cong C_{44},\ G_{15} \cong C_2 \times C_{22},\ |G_{14}| = |G_{15}| = 44;$
$G_{16} = \langle a, b \mid a^4 = b^{11} = 1,\ a^{-1}ba = b^{-1}\rangle = \langle b\rangle \rtimes \langle a\rangle \cong C_{11} \rtimes C_4,\ |G_{16}| = 44;$
$G_{17} \cong D_{44},\ |G_{17}| = 44;$
$G_{18} \cong C_{45},\ G_{19} \cong C_{15} \times C_3,\ |G_{18}| = |G_{19}| = 45;$
$G_{20} \cong C_{46},\ G_{21} \cong D_{46},\ |G_{20}| = |G_{21}| = 46;$
$G_{22} \cong C_{47},\ |G_{22}| = 47.$

In view of Lemmas 2.11, 2.12, 2.13, 2.10, and 2.15, the groups $G_1$–$G_7$ and $G_9$–$G_{22}$ are determined by their endomorphism monoids in the class of all groups.

Let us consider the group

$$G_8 = \langle a, b, c \mid a^2 = b^3 = c^7 = (ac)^2 = b^{-1}aba = c^{-1}b^{-1}cbc^{-1} = 1\rangle.$$

The group $G_8$ can be presented as follows:

$$\begin{aligned} G_8 &= \langle a, b, c \mid a^2 = b^3 = c^7 = 1,\ ab = ba,\ a^{-1}ca = c^{-1},\ b^{-1}cb = c^2\rangle \\ &= \langle c\rangle \rtimes (\langle a\rangle \times \langle b\rangle) \cong C_7 \rtimes (C_2 \times C_3) \end{aligned}$$

or

$$G_8 = \langle d, c \mid d^6 = c^7 = 1,\ d^{-1}cd = c^5\rangle = \langle c\rangle \rtimes \langle d\rangle \cong C_7 \rtimes C_6.$$

By Lemma 2.15, the group $G_8$ is determined by its endomorphism monoid in the class of all groups.

We have proved

**Theorem 3.1.** *The finite groups of orders 37–39 and 41–47 are determined by their endomorphism monoids in the class of all groups.*

## 4. GROUPS OF ORDER 40

According to [24], there exist 14 pairwise non-isomorphic groups of order 40:
$\mathcal{G}_1 = C_{40};$
$\mathcal{G}_2 = C_{20} \times C_2;$
$\mathcal{G}_3 = C_{10} \times C_2 \times C_2;$
$\mathcal{G}_4 = \langle a, b \mid a^8 = b^5 = 1,\ a^{-1}ba = b^{-1}\rangle \cong C_5 \rtimes C_8;$

$\mathcal{G}_5 = \langle a, b \mid a^8 = b^5 = a^{-1}b^2ab = b^{-1}a^{-1}bab^{-1} = b^{-1}a^4ba^4 = 1\rangle$
$\quad = \langle a, b \mid a^8 = b^5 = 1,\ a^{-1}ba = b^2\rangle \cong C_5 \rtimes C_8;$
$\mathcal{G}_6 = \langle a, b, c \mid b^2a^2 = baba^{-1} = a^4 = a^{-1}cac = c^{-1}b^{-1}cb = c^5$
$\quad = (a^{-1}c)^2a^{-2} = 1\rangle;$
$\mathcal{G}_7 = C_4 \times D_{10};$
$\mathcal{G}_8 = D_{40};$
$\mathcal{G}_9 = \langle a, b, c \mid a^4 = b^2 = c^5 = 1,\ ab = ba,\ bc = cb,\ a^{-1}ca = c^{-1}\rangle$
$\quad = \langle b\rangle \times (\langle c\rangle \rtimes \langle a\rangle) \cong C_2 \times (C_5 \rtimes C_4);$
$\mathcal{G}_{10} = \langle a, b, c \mid a^2 = b^2 = c^5 = (ac)^2 = c^{-1}bcb = (ba)^4 = 1\rangle;$
$\mathcal{G}_{11} = C_5 \times D_8;$
$\mathcal{G}_{12} = C_5 \times Q;$
$\mathcal{G}_{13} = \langle a, b, c \mid a^4 = b^2 = c^5 = 1,\ ab = ba,\ bc = cb,\ a^{-1}ca = c^2\rangle \cong C_2 \times (C_5 \rtimes C_4);$
$\mathcal{G}_{14} = C_2 \times C_2 \times D_{10}.$

Lemmas 2.11, 2.13, 2.10, 2.15, and 2.14 imply the following theorem

**Theorem 4.1.** *The groups $\mathcal{G}_1$–$\mathcal{G}_5$, $\mathcal{G}_7$–$\mathcal{G}_9$, and $\mathcal{G}_{11}$–$\mathcal{G}_{14}$ are determined by their endomorphism monoids in the class of all groups.*

We consider the groups $\mathcal{G}_6$ and $\mathcal{G}_{10}$ in the next two sections.

## 5. GROUP $\mathcal{G}_6$

Let us consider the group

$$\mathcal{G}_6 \quad = \quad \langle a, b, c \mid b^2a^2 = baba^{-1} = a^4 = a^{-1}cac = c^{-1}b^{-1}cb = c^5 = (a^{-1}c)^2a^{-2} = 1\rangle.$$

The group $\mathcal{G}_6$ can be presented as follows:

$$\begin{aligned} \mathcal{G}_6 &= \langle a, b, c \mid a^4 = c^5 = 1,\ a^2 = b^2,\ a^{-1}ca = c^{-1},\ bc = cb,\ b^{-1}ab = a^{-1}\rangle \\ &= \langle a, b, c \mid a^4 = c^5 = 1,\ a^2 = b^2,\ c^{-1}ac = ac^2,\ bc = cb,\ b^{-1}ab = a^{-1}\rangle \\ &= \langle c\rangle \rtimes \langle a, b\rangle = C_5 \rtimes Q, \\ C_5 &= \langle c\rangle,\ \ Q = \langle a, b\rangle = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1}\rangle. \end{aligned}$$

In this section, we shall prove the following theorem.

**Theorem 5.1.** *A finite group $G$ is isomorphic to $\mathcal{G}_6$ if and only if there exists $x \in I(G)$ such that the following properties hold:*
$1^0$ $K(x) \cong \mathrm{End}(Q)$;
$2^0$ $H(x) = \{0\}$;
$3^0$ $|[x]| = 5$;
$4^0$ $J(x) = \{0\}$;
$5^0$ $|V(x)| = 4 \cdot 5$;
$6^0$ $P(x) \cong \mathrm{End}(C_5)$;
$7^0$ $K_0(y) = K_0(x)$ *for each* $y \in [x]$.

*Proof. Necessity.* Let $G = \mathcal{G}_6$ and $G$ be given by the generating relations as presented above. Denote by $x$ the projection of $G$ onto its subgroup $Q = \langle a, b\rangle$. Then $\mathrm{Im}\, x = \langle a, b\rangle$ and $\mathrm{Ker}\, x = \langle c\rangle$. We have to prove that $x$ satisfies properties $1^0$–$7^0$.

Property $1^0$ follows from Lemma 2.3. Since $|Q|$ and $|\langle c\rangle|$ are coprime, Lemma 2.4 implies property $2^0$.

The set $[x]$ consists of the idempotents of $\mathrm{End}(G)$ such that $yx = x$ and $xy = y$. Hence $a^{-1}\cdot ay,\, b^{-1}\cdot by \in \mathrm{Ker}\, x,\ cy = 1$, and

$$ay = ac^i,\ by = bc^j,\ cy = 1 \tag{5.1}$$

for some $i, j \in \mathbb{Z}_5$. The map $y$ given by (5.1) preserves the generating relations of $G$ and can be extended to an endomorphism of $G$ if and only if $i \in \mathbb{Z}_5$ and $j = 0$. Each such endomorphism is an idempotent. Therefore, $|\,[x]| = 5$ and property $3^0$ holds.

By Lemma 2.5, $J(x)$ consists of the endomorphisms $y$ of $G$, where

$$ay = by = 1,\ cy = c^i,\ i \in \mathbb{Z}_5. \tag{5.2}$$

The map $y$ given by (5.2) preserves the generating relations of $G$ and can be extended to an endomorphism of $G$ if and only if $i = 0$, i.e., $y = 0$. Hence property $4^0$ is true.

The set $V(x)$ consists of the automorphisms $y$ of $G$ such that $yx = x$, i.e., $g^{-1}\cdot gx \in \mathrm{Ker}\, x = \langle c\rangle$ for each $g \in G$ and

$$ay = ac^i,\ by = bc^j,\ cy = c^k \tag{5.3}$$

for some $i, j, k \in \mathbb{Z}_5$. The map $y$ given by (5.3) preserves the generating relations of $G$ and can be extended to an endomorphism of $G$ if and only if $j = 0$. This endomorphism is an automorphism if and only if $k$ and 5 are coprime. Hence $|V(x)| = 4\cdot 5$ and property $5^0$ holds.

By Lemma 2.7, $P(x)$ consists of the endomorphisms $y$ of $G$ such that

$$ay = a,\ by = b,\ cy = c^i,\ i \in \mathbb{Z}_5. \tag{5.4}$$

The map $y$ given by (5.4) preserves the generating relations of $G$ and can be extended to an endomorphism of $G$ for each $i \in \mathbb{Z}_5$. It follows from here that the submonoid $P(x)$ of $\mathrm{End}(G)$ is isomorphic to $\mathrm{End}(C_5)$, i. e., $x$ satisfies property $6^0$.

It was proved that $[x]$ consists of maps $y_i,\ i \in \mathbb{Z}_5$, where

$$ay_i = ac^i,\ by = b,\ cy = 1.$$

By Lemmas 2.2 and 2.3,

$$\mathrm{Im}\, y_i = \langle ac^i, b\rangle \cong \mathrm{Im}\, x = Q,\ K(y_i) \cong \mathrm{End}(Q).$$

In view of [13], Lemma 1 and Theorem 14, the nilpotent elements of $K(y_i)$ consist of maps $z_{jk}$ $(j, k \in \mathbb{Z}_2)$, where

$$(ac^i)z_{jk} = (ac^i)^{2j} = a^{2j},\ bz_{jk} = (ac^i)^{2k} = a^2,\ cz_{jk} = 1,$$

i.e.,

$$az_{jk} = (ac^i)^{2j} = a^{2j},\ bz_{jk} = (ac^i)^{2k} = a^2,\ cz_{jk} = 1.$$

It follows that $K_0(y) = K_0(x)$ for each $y \in [x]$ and property $7^0$ is true. The necessity is proved.

*Sufficiency.* Let $G$ be a finite group such that there exists $x \in I(G)$ which satisfies properties $1^0$–$7^0$ of the theorem. By property $3^0$, $x$ is non-trivial ($x \notin \{0, 1\}$).

By Lemma 2.1,

$$G = \mathrm{Ker}\, x \rtimes \mathrm{Im}\, x. \tag{5.5}$$

Denote $M = \mathrm{Ker}\, x$. The semidirect product (5.5) is not a direct product, because otherwise the projection of $G$ onto its subgroup $\mathrm{Ker}\, x$ is a non-trivial element in $J(x)$ which contradicts property $4^0$.

Lemma 2.3 and property $1^0$ imply

$$\mathrm{End}(\mathrm{Im}\, x) \cong \mathrm{End}(Q).$$

Since the quaternion group is determined by its endomorphism monoid in the class of all groups (Lemma 2.14), we have $\operatorname{Im} x \cong Q$ and

$$\operatorname{Im} x = Q = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1}\rangle$$

(we identified $\operatorname{Im} x$ and $Q$).

Lemma 2.4 and property $2^0$ imply that $\operatorname{Ker} x$ is a $2'$-group, i.e., $\operatorname{Im} x$ is a Sylow 2-subgroup of $G$. Since the Sylow 2-subgroups of $G$ are conjugate, Lemma 2.1 and property $3^0$ imply that there exist 5 Sylow 2-subgroups of $G$ and

$$[M : N_M(\operatorname{Im} x)] = 5. \tag{5.6}$$

In view of (5.6) and property $5^0$, we have

$$|\widehat{M}| = 5,\ M/(M \cap Z(G)) \cong C_5. \tag{5.7}$$

It follows that all $5'$-elements of $M$ belong into the centre of $G$ and

$$M = M_5 \times M_{5'},\ M_{5'} \subset Z(G),$$

where $M_5$ and $M_{5'}$ are a Sylow 5-subgroup and a Hall $5'$-subgroup of $G$, respectively. Hence

$$G = M_{5'} \times (M_5 \leftthreetimes \operatorname{Im} x).$$

Denote by $\pi$ the projection of $G$ onto its subgroup $M_{5'}$. Then $\pi \in J(x)$ and, by property $4^0$, $\pi = 0$. Therefore, $M_{5'} = \langle 1\rangle$, $M = M_5$, and $M = \operatorname{Ker} x$ is a 5-group. By (5.7), $M$ is an Abelian 5-group.

Let us consider the map

$$y_i : \begin{cases} a \longmapsto a, \\ b \longmapsto b, \\ h \longmapsto h^i,\ h \in M = \operatorname{Ker} x, \end{cases}$$

where $i$ is an integer. Since $M$ is Abelian, it is easy to check that $y_i$ can be uniquely extended to an endomorphism of $G$ and $y_i \in P(x)$. Clearly, $y_i y_j = y_{i\cdot j}$ for each integer $i$ and $j$. Therefore, property $6^0$ implies that

$$h^5 = 1 \text{ for each } h \in M$$

and $M$ is an elementary Abelian 5-group. In view of (5.7), there exist $c \in M$ such that

$$M = \operatorname{Ker} x = (M \cap Z(G)) \times \langle c\rangle,\ \langle c\rangle \cong C_5,\ c \notin M \cap Z(G).$$

Hence

$$G = ((M \cap Z(G)) \times \langle c\rangle) \leftthreetimes Q = ((M \cap Z(G)) \times \langle c\rangle) \leftthreetimes \langle a, b\rangle. \tag{5.8}$$

We can assume that $ac \neq ca$, because the case $bc \neq cb$ can be considered similarly. Therefore, by (5.8), there exist $i \in \mathbb{Z}_5,\ i \neq 0$, and $c_0 \in M \cap Z(G)$ such that

$$c^{-1}ac = ac^i c_0.$$

If $c_0 \neq 1$, there exists $c_1 \in M \cap Z(G)$ such that $c_0 = c_1^i$ and we have

$$\begin{aligned} c^i c_0 &= c^i c_1^i = (cc_1)^i, \\ (cc_1)^{-1} a (cc_1) &= c_1^{-1} \cdot c^{-1}ac \cdot c_1 = c^{-1}ac = ac^i c_0 = ac^i c_1^i = a(cc_1)^i, \end{aligned}$$

i.e., we can replace the element $c$ by the element $cc_1$. It follows that we can assume that

$$c^{-1}ac = ac^i,\ i \neq 0. \tag{5.9}$$

In view of (5.7) and property $3^0$,

$$[x] = \{x\widehat{c}^j \mid j \in \mathbb{Z}_5\}.$$

Choose $j \in \mathbb{Z}_5$ and denote $y = x\widehat{c}^j$. The nilpotent endomorphisms of $Q = \langle a, b\rangle$ are the maps

$$a \longmapsto a^{2k},\ b \longmapsto a^{2l};\ k, l \in \mathbb{Z}_2$$

by Lemma 1 and Theorem 14 in [12]. Therefore, the nilpotent endomorphisms of $K(y)$ consist of maps $z_{kl}$, where

$$(c^{-j}ac^j)z_{kl} = c^{-j}a^{2k}c^j,\ (c^{-j}bc^j)z_{kl} = c^{-j}a^{2l}c^j,\ hz_{kl} = 1,\ h \in M.$$

Since $c \in M$, we have

$$az_{kl} = c^{-j}a^{2k}c^j,\ bz_{kl} = c^{-j}a^{2l}c^j. \tag{5.10}$$

By property $7^0$, $z_{kl}$ is a nilpotent element of $K(x)$. Therefore,

$$az_{kl} = a^{2k},\ bz_{kl} = a^{2l}. \tag{5.11}$$

It follows from (5.10) and (5.11) that $c^{-j}a^2c^j = a^2$ for each $j \in \mathbb{Z}_5$, i.e.,

$$c^{-1}a^2c = a^2. \tag{5.12}$$

In view of (5.9) and (5.12), we have

$$\begin{aligned} a^{-1}c^{-1}a &= c^{i-1},\ a^{-1}ca = c^{1-i}, \\ a^2 &= c^{-1}a^2c = (ac^i)^2 = a^2 \cdot a^{-1}c^ia \cdot c^i = a^2c^{i(1-i)} \cdot c^i = a^2c^{2i-i^2}, \end{aligned}$$

and $2i - i^2 \equiv 0 \pmod 5$. Since $i \neq 0$, we have $i = 2$ and

$$c^{-1}ac = ac^2.$$

Let us now consider properties of $b$. By (5.8), there exist $s \in \mathbb{Z}_5$ and $c_2 \in M \cap Z(G)$ such that

$$\begin{aligned} &c^{-1}bc = bc^sc_2,\ b^{-1}c^{-1}b = c^{s-1}c_2,\ b^{-1}cb = c^{1-s}c_2^{-1}, \\ &c^{-1}b^2c = bc^s \cdot bc^s \cdot c_2^2 = b^2 \cdot b^{-1}c^sb \cdot c^sc_2^2 \\ &\qquad = b^2 \cdot c^{s(1-s)}c_2^{-s} \cdot c^sc_2^2 = b^2 \cdot c^{2s-s^2}c_2^{2-s}. \end{aligned} \tag{5.13}$$

In view of $a^2 = b^2$, (5.12), and (5.13), we have

$$2s - s^2 \equiv 0 \pmod 5,\ c_2^{2-s} = 1,$$

i.e., $s = 0,\ c_2 = 1$ or $s = 2$. If $s = 0,\ c_2 = 1$, then $bc = cb$ and

$$G = \langle a, b, c \mid a^4 = c^5 = 1,\ a^2 = b^2,\ c^{-1}ac = ac^2,\ bc = cb,\ b^{-1}ab = a^{-1}\rangle \cong \mathscr{G}_6.$$

Assume that $s = 2$. Then

$$\begin{aligned} c^{-1}bc &= bc^2c_2,\ b^{-1}cb = c^{-1}c_2^{-1}, \\ c^{-1}abc &= c^{-1}ac \cdot c^{-1}bc = ac^2 \cdot bc^2c_2 = ab \cdot b^{-1}c^2b \cdot c^2c_2 \\ &= ab \cdot c^{-2}c_2^{-2} \cdot c^2c_2 = ab \cdot c_2^{-1}. \end{aligned}$$

Since $(ab)^4 = 1$, we have $c_2^{-4} = 1,\ c_2 = 1$, and

$$c \cdot ab = ab \cdot c.$$

Denote $b_0 = ab$. It is easy to check that $b_0^2 = a^2,\ b_0^{-1}ab_0 = a^{-1}$, and

$$G = \langle a, b_0, c \mid a^4 = c^5 = 1,\ a^2 = b_0^2,\ c^{-1}ac = ac^2,\ b_0c = cb_0,\ b_0^{-1}ab_0 = a^{-1}\rangle \cong \mathscr{G}_6.$$

We have proved that in all possible cases $G \cong \mathscr{G}_6$. The sufficiency is proved. The theorem is proved.

**Theorem 5.2.** *The group $\mathscr{G}_6$ is determined by its endomorphism monoid in the class of all groups.*

*Proof.* Let $G^*$ be a group such that the endomorphism monoids of $G^*$ and $\mathscr{G}_6$ are isomorphic:

$$\mathrm{End}(G^*) \cong \mathrm{End}(\mathscr{G}_6). \tag{5.14}$$

Denote by $z^*$ the image of $z \in \mathrm{End}(\mathscr{G}_6)$ in isomorphism (5.14). Since $\mathrm{End}(G^*)$ is finite, so is $G^*$ ([1], Theorem 2). By Theorem 5.1, there exists $x \in I(\mathscr{G}_6)$ which satisfy properties $1^0 - 7^0$ of the theorem. In view of isomorphism (5.14), the endomorphism $x^*$ satisfies properties $1^0 - 7^0$, where always $x$ and $y$ are replaced by $x^*$ and $y^*$, respectively. Using now Theorem 5.1 for $G^*$, it follows that $G^*$ and $\mathscr{G}_6$ are isomorphic. The theorem is proved.

## 6. ON ENDOMORPHISMS OF $\mathscr{G}_{10}$

Let us consider the group

$$\mathscr{G}_{10} = \langle a, b, d \mid a^2 = b^2 = d^5 = (ad)^2 = d^{-1}bdb = (ba)^4 = 1\rangle.$$

It follows from defining relations of $\mathscr{G}_{10}$ that

$$a^{-1}da = d^{-1},\ bd = db.$$

Denote $c = aba$. Then

$$\begin{aligned} c^2 &= abaaba = 1,\ 1 = (ba)^4 = babababa = bcbc,\ bc = cb, \\ dc &= daba = a \cdot a^{-1}da \cdot ba = ad^{-1}ba = abd^{-1}a = aba \cdot a^{-1}d^{-1}a = cd. \end{aligned}$$

Therefore,

$$\mathscr{G}_{10} = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^5 = 1,\ bc = cb,\ bd = db,\ dc = cd,\\ a^{-1}da = d^{-1},\ a^{-1}ba = c,\ a^{-1}ca = b\rangle$$

and

$$\begin{aligned} \mathscr{G}_{10} &= \langle b, c, d\rangle \rtimes \langle a\rangle = ((\langle b\rangle \times \langle c\rangle \times \langle d\rangle)) \rtimes \langle a\rangle \\ &= ((\langle b\rangle \times \langle c\rangle)) \rtimes (\langle d\rangle \rtimes \langle a\rangle) = \langle d\rangle \rtimes (((\langle b\rangle \times \langle c\rangle)) \rtimes \langle a\rangle). \end{aligned}$$

Our aim is to prove the following theorem.

**Theorem 6.1.** *A finite group $G$ is isomorphic to $\mathscr{G}_{10}$ if and only if there exist $x, y, z \in I(G)$ which satisfy condition $\mathscr{C}_0(x; y, z)$ and the following properties:*
$1^0$ $K(x) \cong \mathrm{End}(C_2)$;
$2^0$ $K(y) \cong \mathrm{End}(D_{10})$;
$3^0$ $K(z) \cong \mathrm{End}(D_8)$;
$4^0$ $|\{u \in K(z) \mid xu = u,\ ux = 0\}| = 4$.

*Proof. Necessity.* Let $G = \mathscr{G}_{10}$. We have to prove that there exist $x, y, z \in I(G)$ which satisfy condition $\mathscr{C}_0(x; y, z)$ and properties $1^0 - 4^0$ of the theorem.

Denote

$$K = \langle a\rangle,\ G_1 = \langle d\rangle,\ G_2 = \langle b\rangle \times \langle c\rangle,\ H = \langle 1\rangle.$$

Then

$$\begin{aligned} G &= H \leftthreetimes ((G_1 \times G_2) \leftthreetimes K), \\ \langle G_1, K \rangle &= G_1 \leftthreetimes K \cong D_{10},\ \langle G_2, K \rangle = G_2 \leftthreetimes K \cong D_8. \end{aligned}$$

We can now use Lemma 2.17 for the case $n = 2$. By Lemma 2.17, there exist $x, y, z \in I(G)$ $(x_1 = y, x_2 = z)$ which satisfy condition $\mathscr{C}_0(x; y, z)$:

$$\mathrm{Im}\, x = K \cong C_2,\ \mathrm{Im}\, y = G_1 \leftthreetimes K \cong D_{10},\ \mathrm{Im}\, z = G_2 \leftthreetimes K \cong D_8.$$

Lemma 2.3 implies that

$$K(x) \cong \mathrm{End}(C_2),\ K(y) \cong \mathrm{End}(D_{10}),\ K(z) \cong \mathrm{End}(D_8).$$

Hence properties $1^0$, $2^0$, and $3^0$ of the theorem hold. In view of Lemma 2.16 (use it for the groups $\mathrm{Im}\, y$ and $\mathrm{Im}\, z$), property $4^0$ is also true. The necessity is proved.

*Sufficiency.* Let $G$ be a finite group and there exist $x, y, z \in I(G)$ which satisfy property $\mathscr{C}_0(x; y, z)$ and properties $1^0 - 4^0$ of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{10}$.

By Lemma 2.17, $G$ decomposes into the semidirect product

$$G = (G_1 \times G_2) \leftthreetimes K,$$

where

$$\begin{aligned} \mathrm{Im}\, x &= K,\ \mathrm{Im}\, y = G_1 \leftthreetimes K,\ \mathrm{Im}\, z = G_2 \leftthreetimes K, \\ \mathrm{Ker}\, x &= (G_1 \times G_2),\ \mathrm{Ker}\, y = G_2,\ \mathrm{Ker}\, z = G_1. \end{aligned}$$

In view of Lemma 2.3 and properties $1^0$, $2^0$, and $3^0$,

$$\begin{aligned} \mathrm{End}(K) &= \mathrm{End}(\mathrm{Im}\, x) \cong \mathrm{End}(C_2), \\ \mathrm{End}(G_1 \leftthreetimes K) &= \mathrm{End}(\mathrm{Im}\, y) \cong \mathrm{End}(D_{10}), \\ \mathrm{End}(G_2 \leftthreetimes K) &= \mathrm{End}(\mathrm{Im}\, z) \cong \mathrm{End}(D_8). \end{aligned}$$

Since each finite Abelian group and dihedral groups are determined by their endomorphism monoids in the class of all groups (Lemmas 2.11 and 2.13), we have

$$K = \langle a \rangle \cong C_2,\ G_1 \leftthreetimes K \cong D_{10},\ G_2 \leftthreetimes K \cong D_8$$

for some $a \in G$. The isomorphism $G_1 \leftthreetimes K \cong D_{10}$ implies that there exists $d \in G_1$ such that

$$a^{-1} d a = d^{-1}.$$

Let us use Lemma 2.16 for the group $\mathrm{Im}\, z \cong D_8$. By $4^0$, $x$ satisfies conditions $1^0$ and $2^0$ of the lemma. Therefore, there exist $b, c \in \mathrm{Im}\, z \cap \mathrm{Ker}\, x$ such that

$$\mathrm{Im}\, z = (\langle b \rangle \times \langle c \rangle) \leftthreetimes \langle a \rangle = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, bc = cb, a^{-1} b a = c \rangle.$$

We have proved that

$$\begin{aligned} G = {} & \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^5 = 1,\ bc = cb,\ bd = db, \\ & dc = cd,\ a^{-1} d a = d^{-1},\ a^{-1} b a = c,\ a^{-1} c a = b \rangle, \end{aligned}$$

i. e., $G \cong \mathscr{G}_{10}$. The sufficiency is proved.

The theorem is proved.

**Theorem 6.2.** *The group $\mathscr{G}_{10}$ is determined by its endomorphism monoid in the class of all groups.*

The proof of Theorem 6.2 is similar to that of Theorem 5.2.

## 7. CONCLUSIONS

We studied the determinability of groups of orders 37–47 by their endomorphism monoids. We proved that all these groups are determined by their endomorphism monoids in the class of all groups. The technique developed in this paper can be applied to other small groups and it can be implemented in the software GAP.

## ACKNOWLEDGEMENTS

This work was supported by the Estonian Ministry of Education and Research (project SF0140011s09). The publication costs of this article were covered by the Estonian Academy of Sciences.

## REFERENCES

1. Alperin, J. L. Groups with finitely many automorphisms. *Pacific J. Math.*, 1962, **12**, 1–5.
2. Ascione, J. A., Havas, G., and Leedham-Green, C. R. A computer aided classification of certain groups of prime order. *Bull. Austral. Math. Soc.*, 1977, **17**, 257–274.
3. Bors, A. On the dynamics of endomorphisms of finite groups. *AAECC*, 2016, doi:10.1007/s00200-016-0304-9.
4. *Can finite non-isomorphic groups of the same order have isomorphic endomorphism monoids?* (`http://math.stackexchange.com/questions/801585`) (accessed 12 December 2016).
5. Cayley, A. Desiderata and suggestions. No. 1. The theory of groups. *Amer. J. Math.*, 1878, **1**, 50–52.
6. Chu, H., Hoshi, A., Hu, S., and Kang, M. Noether's problem for groups of order 243. *J. Algebra*, 2015, **442**, 233–259.
7. Grigorchuk, R. I. and Mamaghani, M. G. On use of iterates of endomorphisms for constructing groups with specific properties. *Mat. Stud.*, 1997, **8**(2), 198–206.
8. James, R., Newman, M. F., and O'Brien, E. A. The groups of order 128. *J. Algebra*, 1990, **129**, 136–158.
9. Leibak, A. and Puusemp, P. On endomorphisms of groups of order 36. *Proc. Estonian Acad. Sci.*, 2016, **65**, 237–254.
10. O'Brien, E. A. The groups of order 256. *J. Algebra*, 1991, **143**, 219–235.
11. O'Brien, E. A. and Vaughan-Lee, M. R., The groups with order $p^7$ for odd prime $p$. *J. Algebra*, 2005, **292**, 243–258.
12. Puusemp, P. Idempotents of the endomorphism semigroups of groups. *Acta Comment. Univ. Tartuensis*, 1975, **366**, 76–104 (in Russian).
13. Puusemp, P. Endomorphism semigroups of generalized quaternion groups. *Acta Comment. Univ. Tartuensis*, 1976, **390**, 84–103 (in Russian).
14. Puusemp, P. On endomorphism semigroups of symmetric groups. *Acta Comment. Univ. Tartuensis*, 1985, **700**, 42–49 (in Russian).
15. Puusemp, P. An embedding theorem for finite groups. *Proc. Estonian Acad. Sci. Phys. Math.*, 1997, **46**, 235–240.
16. Puusemp, P. A characterization of divisible and torsion Abelian groups by their endomorphism semigroups. *Algebras Groups Geom.*, 1999, **16**, 183–193.
17. Puusemp, P. Characterization of a semidirect product of groups by its endomorphism semigroup. In *Proceedings of the International Conference on Semigroups, Braga, June 18–23, 1999* (Smith, P., Giraldes, E., and Martins, P., eds). World Scientific, Singapore, Inc., 2000, 161–170.
18. Puusemp, P. On the definability of a semidirect product of cyclic groups by its endomorphism semigroup. *Algebras Groups Geom.*, 2002, **19**, 195–212.
19. Puusemp, P. Groups of order 24 and their endomorphism semigroups. *Fundam. Prikl. Mat.*, 2005, **11**(3), 155–172 (in Russian).
20. Puusemp, P. Groups of order less than 32 and their endomorphism semigroups. *J. Nonlinear Math. Phys.*, 2006, **13**, Supplement, 93–101.
21. Puusemp, P. and Puusemp, P. On endomorphisms of groups of order 32 with maximal subgroups $C_4 \times C_2 \times C_2$. *Proc. Estonian Acad. Sci.*, 2014, **63**, 105–120.
22. Puusemp, P. and Puusemp, P. On endomorphisms of groups of order 32 with maximal subgroups $C_8 \times C_2$. *Proc. Estonian Acad. Sci.*, 2014, **63**, 355–371.
23. Puusemp, P. and Puusemp, P. On endomorphisms of groups of order 32 with maximal subgroups $C_2^4$ or $C_4^2$. *Proc. Estonian Acad. Sci.*, 2016, **65**, 1–14.
24. The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.6*; 2014 (`http://www.gap-system.org`) (accessed 12 December 2016).

**Järkudega 37–47 rühmade endomorfismidest**

Alar Leibak ja Peeter Puusemp

Autorite varasemates töödes on kirjeldatud kõik rühmad, mille endomorfismimonoidid on isomorfsed mingi väiksemat kui 37. järku rühma endomorfismimonoididega. Käesolevas artiklis on tõestatud, et lõplikud rühmad järkudega 37–47 on määratud oma endomorfismimonoididega kõigi rühmade klassis. Selleks on mainitud rühmade kirjeldused antud nende endomorfismimonoidide kaudu.