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CONTROL THEORY

Stability and stabilizability of linear time-delay systems on homogeneous time scales

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Abstract. This paper provides necessary and sufficient conditions for the exponential stability of a linear retarded time-delay system defined on a homogeneous time scale. Conditions are formulated in terms of a characteristic equation associated with the system. This approach is then used to develop feedback stabilizability criteria.

Key words: linear control system, time scales, time delay, stability.

1. INTRODUCTION

Traditionally dynamic systems are studied in a continuous time domain. Over the years numerous sophisticated techniques have been developed for an efficient analysis. At the same time the digital world motivates development of rigorous methods for analysis of systems evolving in discrete time, not necessarily uniform. A lot of efforts have been made to adopt the ideas from continuous settings to the discrete domain. However, it is still a great challenge for scientists how to effectively merge the two paradigms. One of the most successful attempts is based on the notion of time scales, which first appeared in [12]. The main idea extends beyond the continuous and discrete domains. The theory of systems on time scales comprises nonuniformly sampled systems, interval models, hybrid systems, etc. This formalism has been successfully applied to solve various problems. A good introduction can be found in [4].

There exists a large class of systems naturally obeying the delay property, e.g. communication systems, long transmission lines, pipelines, remote control systems such as satellites, etc. Some of the available results can be found in classical monographs such as [1,10,11]. The majority of works deal separately with the continuous- or discrete-time cases. Very few results are available for the study of time-delay systems on time scales. To address stability related problems, the approach based on the Lyapunov functions is usually used. In [20] the necessary and sufficient conditions for the asymptotic stability of linear positive systems with bounded time-varying delays were derived. A similar approach was applied in [14], where the stability of delay impulsive (at fixed times) systems on time scales was studied. A slightly larger class of impulsive

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hybrid systems was considered in [19]. Furthermore, [13] proposes sufficient conditions for the practical stability of hybrid dynamic systems. Some more specific adaptations for the case of neural networks are reported in [8].

In this paper, the time scales calculus is used to unify stability studies of linear time-delay systems of retarded type (when delay enters the state) defined on homogeneous time scales. We rely on a functional representation of a time-delay system following the ideas presented in [11,16]. In brief, contributions of this paper can be seen in the following. The Laplace transform is extended to operate with time-delay systems defined on a homogeneous time scale. The developed tools, given in terms of the characteristic equation of a system, are then used to derive necessary and sufficient conditions for the exponential stability. Furthermore, necessary and sufficient conditions for stabilizability are presented. Since the main results partly (in the discrete-time case) rely on the delay-free case, the necessary proofs for the systems without time delay are collected in Appendix A.

The rest of the paper is organized as follows. Section 2 provides a brief overview of the time scales calculus. Section 3 presents a unified stability definition of a linear time-delay system defined on a homogeneous time scale. It is accompanied by the necessary and sufficient exponential stability condition. The developed mathematical tools to operate with the Laplace transform on time scales are presented. The end of Section 3 is devoted to a stabilizability problem.

2. CALCULUS ON TIME SCALES

The following definitions and a general introduction to time scales calculus can be found in [4]. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. This paper is focused on two cases most important for control theory, i.e. the continuous-time case $\mathbb{T}=\mathbb{R}$ and the discrete-time case $\mathbb{T}=\tau\mathbb{Z}:=\{\tau k \mid k\in\mathbb{Z}\}$ for $\tau>0$. For $t\in\mathbb{T}$ the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by $\sigma(t):=\inf\{s\in\mathbb{T}\mid s>t\}$. The *graininess function* $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t):=\sigma(t)-t$. If $\mu(t)\equiv\text{const}$, then a time scale $\mathbb{T}$ is called *homogeneous*. In this paper we assume that the time scale $\mathbb{T}$ is homogeneous. The *delta derivative* of $\xi:\mathbb{T}\to\mathbb{R}$ is denoted by $\xi^{\Delta}(t)$ and the operators *antiderivative* is denoted by $\int\xi(t)\Delta t$. Table 1 illustrates the above-mentioned for two typical cases of $\mathbb{T}$, where id means identity operator.

In order to simplify exposition of the paper, sometimes the time argument t is omitted, so $\xi:=\xi(t)$.

2.1. Exponential stability

Let $\mathbb{T}$ be a time scale unbounded above and $0\in\mathbb{T}$. A matrix $A\in\mathbb{R}^{n\times n}$ is said to be *regressive* with respect to $\mathbb{T}$ when $I+\mu(t)A$ is invertible for all $t\in\mathbb{T}$, where I denotes the identity matrix. Consider the following linear system of delta differential equations on time scales $\mathbb{T}$

$$x^{\Delta}(t)=Ax(t), \tag{1}$$

where $t\in\mathbb{T}$ and $x(t)\in\mathbb{R}^{n}$.

Table 1. Basic types of operators

$\mathbb{T}$	σ	$\xi^{\sigma}(t)$	μ	Δ	$\xi^{\Delta}(t)$	$\int_a^b f(t)\Delta t$
$\mathbb{R}$	id	$\xi(t)$	0	$\dfrac{\mathrm{d}}{\mathrm{d}t}$	$\dfrac{\mathrm{d}\xi(t)}{\mathrm{d}t}$	$\int_{\mathrm{a}}^{\mathrm{b}} f(t)\mathrm{d}t$
$\tau\mathbb{Z}$	σ	$\xi(t+\tau)$	τ	$\dfrac{\sigma-id}{\tau}$	$\dfrac{\xi(t+\tau)-\xi(t)}{\tau}$	$\sum_{k=\frac{a}{\tau}}^{\frac{b}{\tau}-1}\tau f(k\tau)$

Definition 1 ([4]). *Let A be a regressive matrix and $t_0 \in \mathbb{T}$. Then, a function $X : [t_0, \infty) \to \mathbb{R}^{n\times n}$ that satisfies the matrix delta differential equation*

$$X^{\Delta}(t) = AX(t)$$

and the initial condition $X(t_0) = I$, is a solution called matrix exponential function of A at t_0. Its value at $t \in \mathbb{T}$ is denoted by $\mathrm{e}_A(t,t_0)$.

Example 1. If $\mathbb{T} = \mathbb{R}$, then $\mathrm{e}_A(t,t_0) = \mathrm{e}^{A(t-t_0)}$, where $t, t_0 \in \mathbb{R}$. If $\mathbb{T} = \tau\mathbb{Z}$, then $\mathrm{e}_A(t,t_0) = (I + \tau A)^{\frac{t-t_0}{\tau}}$, where $t, t_0 \in \tau\mathbb{Z}$.

Note that the vector function $t \mapsto \mathrm{e}_A(t,t_0)x_0$ is a solution of (1) with the initial condition $x(t_0) = x_0$.

Definition 2 ([17]). *The system (1) is exponentially stable if there exists a constant $\alpha > 0$ such that for every $t_0 \in \mathbb{T}$ there exists $K = K(t_0) > 1$ with $\|\mathrm{e}_A(t,t_0)x(t_0)\| \leq K\mathrm{e}^{-\alpha(t-t_0)}\|x(t_0)\|$ for $t \geq t_0$.*

Theorem 1 ([17]). *Let $\lambda \in \mathbb{C}$. The scalar system*

$$x^{\Delta}(t) = \lambda x(t)$$

is exponentially stable if and only if one of the following conditions is satisfied:

(i) $\gamma(\lambda) := \limsup\limits_{\mathbb{T}\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \lim\limits_{s\to\mu(t)} \frac{\log|1+s\lambda|}{s}\Delta t < 0$;

(ii) for every $T \in \mathbb{T}$ there exists $t \in \mathbb{T}$ with $t > T$ such that $1 + \mu(t)\lambda = 0$,
where we use the convention $\log 0 = -\infty$ in (i).

Definition 3 ([17]). *Define for arbitrary $t_0 \in \mathbb{T}$*

$$\mathscr{S}_{\mathbb{C}}(\mathbb{T}) := \left\{\lambda \in \mathbb{C} : \limsup_{\mathbb{T}\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \lim_{s\to\mu(t)} \frac{\log|1+s\lambda|}{s}\Delta t < 0\right\}$$

and

$$\mathscr{S}_{\mathbb{R}}(\mathbb{T}) := \{\lambda \in \mathbb{R} \mid \forall T \in \mathbb{T} : \exists t \in \mathbb{T}, t > T : 1 + \mu(t)\lambda = 0\}.$$

Then, the set of exponential stability for $\mathbb{T}$ is given by

$$\mathscr{S}(\mathbb{T}) := \mathscr{S}_{\mathbb{C}}(\mathbb{T}) \cup \mathscr{S}_{\mathbb{R}}(\mathbb{T}).$$

Next, the theorem from [17] is adopted, providing the spectral characterization of the region of exponential stability for a system defined on a homogeneous time scale.

Theorem 2 ([17]). *Let $\mathbb{T}$ be a homogeneous time scale. System (1) is exponentially stable if and only if* $\mathrm{spec}(A) \subset \mathscr{S}(\mathbb{T})$.

Proposition 1 ([2]). *For the special cases of homogeneous time scales, the set $\mathscr{S}(\mathbb{T})$ can be described as follows:*

- *Let $\mathbb{T} = \mathbb{R}$. Then, $\mathscr{S}_{\mathbb{R}}(\mathbb{R}) = \emptyset$ and $\mathscr{S}(\mathbb{R}) = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\}$.*
- *Let $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$. Then, $\mathscr{S}_{\mathbb{R}}(\tau\mathbb{Z}) = \{-1/\tau\}$ and $\mathscr{S}(\tau\mathbb{Z}) = \mathscr{B}_{\frac{1}{\tau}}(-1/\tau)$, where $\mathscr{B}_{\frac{1}{\tau}}(-1/\tau)$ denotes the disc with the centre at $(-1/\tau, 0)$ and the radius of $1/\tau$.*

3. MAIN RESULTS

Consider a linear control system defined on a homogeneous time scale $\mathbb{T}$

$$x^{\Delta}(t) = A_0 x(t) + \int_{-h}^{0} [\Delta N(\xi)] x(t+\xi) + Bu(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A_0 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $N(\xi)$ is an $n \times n$ matrix of bounded variation on $[-h,0]_{\mathbb{T}}$ and is left-continuous at 0. To analyse the stability property, consider an autonomous version of system (2)

$$x^{\Delta}(t) = A_0 x(t) + \int_{-h}^{0} [\Delta N(\xi)] x(t+\xi). \tag{3}$$

For $x : \mathbb{T} \to \mathbb{R}^n$, let $x_t : [-h,0]_{\mathbb{T}} \to \mathbb{R}^n$ be defined by $x_t(\theta) = x(t+\theta)$. Let $||x_t||_h = \sup_{[-h,0]_{\mathbb{T}}} ||x_t(\theta)||$.

Definition 4. *System 3 is exponentially stable if there exist $\alpha > 0$ and $K \geq 1$ such that $||x_t||_h \leq K\mathrm{e}^{-\alpha t}||x_0||_h$ for any $t \geq 0$ and any initial function x_0.*

Remark 1. Observe that for $h = 0$ system (3) reduces to (1). Then x_t is identified with $x(t)$ and the norm $||\cdot||_h$ is the standard norm $||\cdot||$ in $\mathbb{R}^n$. Moreover, for $h = 0$ Definition 4 is equivalent to Definition 2 (on homogeneous time scales).

There exist several definitions of the unilateral Laplace transform on time scales, e.g. [4–6,17]. For the case of the bilateral Laplace transform on time scales see [7].

Definition 5 ([5])**.** *Assume that $x : \mathbb{T} \to \mathbb{C}$ is regulated. Then the Laplace transform of x is defined by*

$$\mathscr{L}\{x\}(z) := \int_{0}^{\infty} x(t) \mathrm{e}^{\sigma}_{\ominus z}(t,0)\Delta t \tag{4}$$

for $z \in \mathscr{D}\{x\}$, where $\mathscr{D}\{x\}$ consists of all[1] *complex numbers $z \in \mathbb{C}$ for which the improper integral exists.*

Theorem 3 ([4])**.** *Assume $x : \mathbb{T} \to \mathbb{C}$ is such that x^{Δ} is regulated and $x(0) = 0$. Then*

$$\mathscr{L}\{x^{\Delta}\}(z) = z\mathscr{L}\{x\}(z)$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty} x(t)\mathrm{e}_{\ominus z}(t,0) = 0$.

Let $\xi > 0$. Define the function $g(t)$ as

$$g(t) := \begin{cases} 0, & 0 \leq t < \xi, \\ x(t-\xi), & t \geq \xi. \end{cases}$$

Proposition 2. *The Laplace transform of the function g is*

$$\mathscr{L}\{g\}(z) = \mathrm{e}_{\ominus z}(\xi,0)\mathscr{L}\{x\}(z).$$

Proof. The proof is based on direct application of Definition 5, i.e.

$$\mathscr{L}\{g\}(z) = \int_{0}^{\infty} g(t)\mathrm{e}^{\sigma}_{\ominus z}(t,0)\Delta t = \int_{\xi}^{\infty} x(t-\xi)\mathrm{e}^{\sigma}_{\ominus z}(t,0)\Delta t.$$

[1] See [4] for the definition of the operator $\ominus$.

According to (ii) and (iii) from [4, Theorem 2.36], the right-hand side of the above equation can be rewritten as

$$\int_{\xi}^{\infty} x(t-\xi)\frac{\mathrm{e}_{\ominus z}(t,0)}{1+\mu z}\Delta t.$$

Define $u := t-\xi$ or equivalently $t = u+\xi$. Then $u \in \widetilde{\mathbb{T}} := \mathbb{T}-\xi$. Since $\mathbb{T}$ is assumed to be a homogeneous time scale, using part (v) of [4, Theorem 2.36] and the fact from [4, Exercise 2.51] (see Appendix B for the proof), we get

$$\int_{0}^{\infty} x(u)\frac{\mathrm{e}_{\ominus z}(u+\xi,0)}{1+\mu z}\widetilde{\Delta} u = \int_{0}^{\infty} x(u)\mathrm{e}^{\sigma}_{\ominus z}(u,0)\mathrm{e}_{\ominus z}(\xi,0)\widetilde{\Delta} u = \mathrm{e}_{\ominus z}(\xi,0)\mathscr{L}\{x\}(z),$$

where $\int_0^\infty \cdots \widetilde{\Delta} u$ means the delta integral on $\widetilde{\mathbb{T}}$. □

Applying the Laplace transform to both sides of (3) and using Theorem 3 and Proposition 2, we get

$$\begin{aligned}&\mathscr{L}\{x^{\Delta}(t)\}(z) - \mathscr{L}\left\{A_0 x(t) + \int_{-h}^{0}[\Delta N(\xi)]x(t+\xi)\right\}(z)\\ &\qquad = z\mathscr{L}\{x\}(z) - A_0\mathscr{L}\{x\}(z) - \int_{-h}^{0}[\Delta N(\xi)]\mathrm{e}_{\ominus z}(-\xi,0)\mathscr{L}\{x\}(z) = 0,\end{aligned}$$

whose characteristic equation can be written as

$$\chi_s(z) := \det\left(zI_n - A_0 - \int_{-h}^{0}[\Delta N(\xi)]\mathrm{e}_{\ominus z}(-\xi,0)\right) = 0, \tag{5}$$

where I_n is the $n\times n$ identity matrix.

Theorem 4. *System (3) is exponentially stable if and only if all the solutions of the characteristic equation $\chi_s(z) = 0$ lie in $\mathscr{S}(\mathbb{T})$.*

To prove the theorem it is necessary to consider two separate cases: continuous (which is proved in [16]) and discrete, which is addressed in the following subsection.

3.1. Uniform discrete-time case

To prove Theorem 4 in the discrete-time case we need several technical results. Recall that the classical $\mathscr{Z}$-transform is defined as

$$\mathscr{Z}\{x\}(z) := \sum_{k=0}^{\infty}\frac{x(k)}{z^k}, \tag{6}$$

where $x : \mathbb{N}_0 \to \mathbb{C}$. Note that Eq. (6) can be extended on $\tau\mathbb{Z}$ as

$$\mathscr{Z}\{x\}(z) = \sum_{k=0}^{\infty}\frac{x(k\tau)}{z^k}. \tag{7}$$

The following proposition defines the relation between the $\mathscr{Z}$-transform and the Laplace transform on a homogeneous discrete-time scale.

Proposition 3. *Let $\mathbb{T} = \tau\mathbb{Z}$ and $x : \mathbb{T} \to \mathbb{C}$ be regulated. Then*

$$\mathscr{L}\{x\}(z) = \tau\frac{\mathscr{Z}\{x\}(1+\tau z)}{1+\tau z}. \tag{8}$$

Proof. According to (ii) and (iii) from [4, Theorem 2.36], (4) can be rewritten as

$$\begin{aligned}\mathscr{L}\{x\}(z) &= \int_0^\infty x(t)\mathrm{e}^\sigma_{\ominus z}(t,0)\Delta t = \int_0^\infty x(t)[1+\mu(t)(\ominus z)]\mathrm{e}_{\ominus z}(t,0)\Delta t \\ &= \int_0^\infty x(t)\frac{\mathrm{e}_{\ominus z}(t,0)}{1+\mu(t)z}\Delta t = \int_0^\infty x(t)\frac{1}{(1+\mu(t)z)\mathrm{e}_z(t,0)}\Delta t. \end{aligned} \tag{9}$$

Using the definition of the improper integral on $\tau\mathbb{Z}$ (see [4, Theorem 1.79]), (9) becomes

$$\mathscr{L}\{x\}(z) = \tau\sum_{k=0}^\infty \frac{x(\tau k)}{(1+\tau z)(1+\tau z)^k} = \frac{\tau}{1+\tau z}\sum_{k=0}^\infty \frac{x(\tau k)}{(1+\tau z)^k} \tag{10}$$

for those values of $z \neq -1/\tau$ for which this series converges. Next, using [4, p. 118] and (7), relation (10) yields (8), which concludes the proof. □

Remark 2. The results of [5] follow as a special case from Proposition 3. In addition, note that (8) coincides with that obtained in [3, Definition 4.1], but the proof is different.

Lemma 1. *The Laplace transform of a function $g(t) = x(t-h)$ on $\mathbb{T} = \tau\mathbb{Z}$ with $h = \tau r$, $r \in \mathbb{Z}^+$ is equal to*

$$\mathscr{L}\{g\}(z) = \frac{1}{(1+\tau z)^{\frac{h}{\tau}}}\mathscr{L}\{x\}(z).$$

Proof. The proof follows from Proposition 2 by observing that $\mathrm{e}_z(t,0) = (1+\tau z)^{\frac{t}{\tau}}$ on $\tau\mathbb{Z}$. However, further an alternative proof is presented. Using relation (8) and the fact that $x(t) = 0$ for $t < 0$, one can write

$$\begin{aligned}\mathscr{L}\{g\}(z) = \tau\frac{\mathscr{Z}\{x(\tau(k-r))\}(1+\tau z)}{1+\tau z} \frac{\tau}{1+\tau z}\sum_{k=0}^\infty \frac{x(\tau k-\tau r)}{(1+\tau z)^k} \\ = \frac{\tau}{1+\tau z}\sum_{k=r}^\infty \frac{x(\tau k-\tau r)}{(1+\tau z)^{k-r}}\cdot\frac{1}{(1+\tau z)^r} \\ = \frac{1}{(1+\tau z)^r}\cdot\frac{\tau}{1+\tau z}\sum_{k=0}^\infty \frac{x(\tau k)}{(1+\tau z)^k} \\ = \frac{1}{(1+\tau z)^{\frac{h}{\tau}}}\mathscr{L}\{x\}(z).\end{aligned}$$

□

For $\mathbb{T} = \tau\mathbb{Z}$, according to [15, Corollary 3.4], system (3) can be represented as

$$x^\Delta(t) = A_0x(t) + \sum_{k=1}^r A_kx(t-k\tau), \tag{11}$$

where $t \geq h := r\tau$, $\tau > 0$, $r \in \mathbb{Z}^+$, and $A_k := N((-k+1)\tau) - N(-k\tau)$. Define $\psi_i(t) := x(t-(r+1-i)\tau)$ and consider the system

$$\begin{aligned}\psi_1^\Delta(t) &= \frac{\psi_2(t)-\psi_1(t)}{\tau} \\ &\vdots \\ \psi_r^\Delta(t) &= \frac{\psi_{r+1}(t)-\psi_r(t)}{\tau} \\ \psi_{r+1}^\Delta(t) &= A_r\psi_1(t)+A_{r-1}\psi_2(t)+A_{r-2}\psi_3(t)+\cdots+A_1\psi_r(t)+A_0\psi_{r+1}(t),\end{aligned} \tag{12}$$

whose characteristic polynomial is defined as

$$\chi_e(z) := \det\left(zI_{(r+1)n} - A_e\right), \tag{13}$$

where

$$A_e := \begin{bmatrix} -\frac{1}{\tau}I & \frac{1}{\tau}I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{1}{\tau}I & \frac{1}{\tau}I \\ A_r & A_{r-1} & A_{r-2} & \cdots & A_1 & A_0 \end{bmatrix}.$$

Theorem 5. *The characteristic equations $\chi_s(z) = 0$ of (11) and $\chi_e(z) = 0$ of (12) are equivalent*[2] *on $\mathbb{T} = \tau\mathbb{Z}$.*

Proof. Consider system (12). According to (13), the characteristic equation can be found as

$$\begin{aligned} \chi_e(z) &= \det(zI - A_e) \\ &= \det\left(\begin{bmatrix} \left(z+\frac{1}{\tau}\right)I & -\frac{1}{\tau}I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \left(z+\frac{1}{\tau}\right)I & -\frac{1}{\tau}I \\ -A_r & -A_{r-1} & -A_{r-2} & \cdots & -A_1 & zI - A_0 \end{bmatrix}\right) = 0, \end{aligned}$$

which, due to the specific structure of the matrix, transforms to

$$\begin{aligned} \det\Bigg(\left(z+\frac{1}{\tau}\right)^r (zI - A_0) - \left(z+\frac{1}{\tau}\right)^{r-1} A_1 \frac{1}{\tau} - \cdots \\ - \left(z+\frac{1}{\tau}\right) A_{r-1}\frac{1}{\tau^{r-1}} - A_r\frac{1}{\tau^r}\Bigg) = 0. \end{aligned} \tag{14}$$

Multiply both sides of (14) by $\tau^r/(1+\tau z)^r$ to get

$$\det\Big(zI - A_0 - A_1(1+\tau z)^{-1} - \cdots - A_{r-1}(1+\tau z)^{-r+1} - A_r(1+\tau z)^{-r}\Big) = 0.$$

Now it is necessary to show that the obtained characteristic equation is equivalent to (5) specified on $\tau\mathbb{Z}$. Recall that (5) is given by

$$\det\left(zI_n - A_0 - \int_{-h}^{0} [\Delta N(\xi)] \mathrm{e}_{\ominus z}(-\xi, 0)\right) = 0.$$

Since $\mathbb{T} = \tau\mathbb{Z}$, using [15, Corollary 3.4] and the fact that $\mathrm{e}_{\ominus z}(-\xi,0) = 1/(1+\tau z)^{\frac{-\xi}{\tau}}$, Eq. (5) can be rewritten as

$$\det\left(zI_n - A_0 - \tau\sum_{k=-\frac{h}{\tau}}^{\frac{0}{\tau}-1}\left[\frac{N(k\tau+\tau) - N(k\tau)}{\tau}\right]\frac{1}{(1+\tau z)^{\frac{-\xi}{\tau}}}\right) = 0$$

or using $A_{-k} := N(k\tau+\tau) - N(k\tau)$, $\xi = \tau k$, and $h = \tau r$ as

$$\det\left(zI_n - A_0 - \sum_{k=-r}^{-1} A_{-k}(1+\tau z)^k\right) = 0. \tag{15}$$

The same result can be obtained by direct application of Lemma 1to system (11). Observe that (13) and (15) coincide. Therefore, $\chi_e(z) = 0$ and $\chi_s(z) = 0$ are equivalent on $\tau\mathbb{Z}$. □

[2] Equivalence has to be understood in the sense that both (11) and (12) have the same solutions.

Proposition 4. *System (12) is exponentially stable if and only if* $\operatorname{spec}(A_e) \subset \mathscr{S}(\mathbb{T})$.

Proof. The proof follows immediately from Theorem 2. □

Proposition 5. *System (11) is exponentially stable if and only if (12) is exponentially stable.*

Proof. Observe that x satisfies (11) if and only if $\psi = (\psi_1, \ldots, \psi_{r+1})^{\mathrm{T}}$ defined by $\psi_k(t) := x(t-(r-k+1)\tau)$ satisfies (12). If (11) is exponentially stable, then there exist $\alpha > 0$ and $K \geq 1$ such that $||x_t||_h \leq K\mathrm{e}^{-\alpha t}||x_0||_h$. Observe that $||\psi(t)|| \leq \sqrt{r+1}||x_t||_h$ and $||x_t||_h \leq ||\psi(t)||$. Thus, $||\psi(t)|| \leq \sqrt{r+1}K\mathrm{e}^{-\alpha t}||x_0||_h \leq \sqrt{r+1}K\mathrm{e}^{-\alpha t}||\psi(0)||$, which means that (12) is exponentially stable. On the other hand, if (12) is stable, then there exist $\alpha > 0$ and $K \geq 1$ such that $||\psi(t)|| \leq K\mathrm{e}^{-\alpha t}||\psi(0)||$. Using the above estimates, we get $||x_t||_h \leq K\mathrm{e}^{-\alpha t}\sqrt{r+1}||x_0||_h$. Therefore, (11) is also exponentially stable. □

Now we are ready to prove Theorem 4 in the uniform discrete-time case.

Proof of Theorem 4. Let first $\mathbb{T} = \tau\mathbb{Z}$. From Proposition 4 it follows that the extended system (12) is exponentially stable if and only if $\operatorname{spec}(A_e) \subset \mathscr{S}(\mathbb{T})$ or equivalently if the solutions of the characteristic equation $\chi_e(z) = 0$ lie in $\mathscr{S}(\mathbb{T})$. Then, according to Theorem 5, both characteristic equations, $\chi_e(z) = 0$ and $\chi_s(z) = 0$, are equivalent on $\tau\mathbb{Z}$. Therefore, by Proposition 5, the exponential stability of (11) means the exponential stability of (12). This concludes the proof of Theorem 4 for $\tau\mathbb{Z}$. Finally, recall that for $\mathbb{T} = \mathbb{R}$ this equivalence was shown in [16]. □

3.2. Stabilizability of linear time-delay systems

Recall several facts of linear systems without delay defined on a homogeneous time scale $\mathbb{T}$

$$x^{\Delta} = Ax + Bu. \tag{16}$$

Definition 6. *System (16) is stabilizable if there is a matrix K such that the system $x^{\Delta} = (A - BK)x$ is exponentially stable.*

The pair (A,B) is called *controllable* if $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$. The following theorem was stated in [2] without a proof.

Theorem 6 ([2])**.** *Let $\mathbb{T}$ be an arbitrary time scale with* $\sup\mathbb{T} = \infty$ *and nonempty stability set $\mathscr{S}(\mathbb{T})$. System (16) is stabilizable if and only if for every $\lambda \notin \mathscr{S}(\mathbb{T})$,* $\operatorname{rank}[\lambda I - A, B] = n$.

For the completeness of the paper a proof of Theorem 6 is given in Appendix A. Recall that the time-delay system defined on a homogeneous time scale $\mathbb{T}$ is given by (2) as

$$x^{\Delta}(t) = A_0 x(t) + \int_{-h}^{0} [\Delta N(\xi)] x(t+\xi) + Bu(t).$$

Definition 7. *System (2) is said to be stabilizable if there exists a feedback of the form $u(t) = -K_0 x(t) - \int_{-h}^{0}[\Delta K(\xi)]x(t+\xi)$ with K being bounded variation on $[-h,0]_{\mathbb{T}}$ and left-continuous at 0 such that the closed-loop system*

$$x^{\Delta}(t) = (A_0 - BK_0)x(t) + \int_{-h}^{0} \Big\{\Delta N(\xi) - B[\Delta K(\xi)]\Big\} x(t+\xi) \tag{17}$$

is exponentially stable.

Remark 3. Observe that if $h = 0$, then Definition 7 reduces to Definition 6.

Lemma 2. *Let $\mathbb{T} = \tau\mathbb{Z}$. The system*

$$x^{\Delta}(t) = A_0 x(t) + \sum_{k=1}^{r} A_k x(t - k\tau) + Bu(t)$$

with $A_k := N((-k+1)\tau) - N(-k\tau)$ is stabilizable if and only if the system

$$\begin{aligned}
\psi_1^{\Delta}(t) &= \frac{\psi_2(t) - \psi_1(t)}{\tau} \\
&\vdots \\
\psi_r^{\Delta}(t) &= \frac{\psi_{r+1}(t) - \psi_r(t)}{\tau} \\
\psi_{r+1}^{\Delta}(t) &= A_r \psi_1(t) + A_{r-1}\psi_2(t) + \cdots + A_1 \psi_r(t) + A_0 \psi_{r+1}(t) + Bu(t)
\end{aligned}$$

is stabilizable.

Proof. Assume that system (2) is stabilizable, i.e. by Definition 7 there exists a feedback of the form $u(t) = -K_0 x(t) - \int_{-h}^{0} [\Delta K(\xi)] x(t+\xi)$ such that the closed-loop system (17) is exponentially stable. Let $\mathbb{T} = \tau\mathbb{Z}$. Then the stabilizing feedback becomes $u(t) = -F_0 x(t) - \sum_{k=1}^{r} F_k x(t - k\tau)$, where $h = r\tau$, $F_k := K((-k+1)\tau) - K(-k\tau)$, $F_0 := K_0$. Using new coordinates $\psi = (\psi_1, \ldots, \psi_{r+1})^{\mathrm{T}}$, $\psi_k(t) := x(t - (r-k+1)\tau)$, the system

$$x^{\Delta}(t) = (A_0 - BF_0)x(t) + \sum_{k=1}^{r} (A_k - BF_k) x(t - k\tau)$$

can be rewritten as

$$\psi^{\Delta} = A_e \psi + B_e u \tag{18}$$

or equivalently as

$$\psi^{\Delta} = (A_e - B_e F)\psi, \tag{19}$$

where

$$A_e := \begin{bmatrix} -\frac{1}{\tau} I & \frac{1}{\tau} I & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{\tau} I & \frac{1}{\tau} I \\ A_r & A_{r-1} & A_{r-2} & A_{r-3} & \cdots & A_1 & A_0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix},$$

and $u := F\psi$ with $F = [F_r, \ldots, F_1, F_0]$.

By Proposition 5 system (19) is exponentially stable, because we assumed stabilizability of system (2), which by Definition 6 implies exponential stability. Hence, the feedback $u = F\psi$ indeed stabilizes (19). The presented arguments can be applied in the opposite way to finalize the proof. Finally, it should be mentioned that the continuous-time case is addressed in [9] and [16]. □

Theorem 7. *System 2 is stabilizable on a homogeneous time scale $\mathbb{T}$ if and only if*

$$\operatorname{rank}\left[zI - A_0 - \int_{-h}^{0} [\Delta N(\xi)] \mathrm{e}_{\ominus z}(-\xi, 0), B \right] = n \tag{20}$$

for $z \notin \mathscr{S}(\mathbb{T})$.

Proof. Let $\mathbb{T} = \tau\mathbb{Z}$ and $z \notin \mathscr{S}(\mathbb{T})$. Start by showing the equivalence of the condition in Theorem 6 specified for system (18) and that in Theorem 7. Now (20) reads as

$$\operatorname{rank}\left[zI - A_0 - \sum_{k=-r}^{-1} A_{-k}(1 + \tau z)^k, B \right] = n \tag{21}$$

with $A_{-k} := N(k\tau+\tau) - N(k\tau)$, $\xi = \tau k$, $h = \tau r$, and the condition stated in Theorem 6 specified for system (18) reads as

$$\operatorname{rank}[zI - A_e, B_e] = n(r+1). \tag{22}$$

Suppose (21) does not hold. This means that there exists $v \neq 0$ such that

$$v^{\mathrm{T}} \left[zI - A_0 - \sum_{k=-r}^{-1} A_{-k}(1+\tau z)^k, B \right] = 0 \tag{23}$$

holds. Then, taking $w^{\mathrm{T}} = [w_0^{\mathrm{T}}, \ldots, w_{r-1}^{\mathrm{T}}, v^{\mathrm{T}}] \neq 0$, where elements are defined as $w_l^{\mathrm{T}} := \tau \sum_{k=1}^{l+1} v^{\mathrm{T}} A_{r-l+k-1}(1+\tau z)^{-k}$ for $l = 0, \ldots, r-1$, yields $w^{\mathrm{T}}[zI - A_e, B_e] = 0$. Indeed, multiplying $[zI - A_e, B_e]$ by w^{T} from the left yields

$$\left[0, 0, \ldots, 0, v^{\mathrm{T}} \left(zI - A_0 - \sum_{k=-r}^{-1} A_{-k}(1+\tau z)^k \right), v^{\mathrm{T}} B \right],$$

which is 0 by (23). Hence, (22) does not hold.

Assume now that (22) does not hold, i.e. there is $w = [w_0^{\mathrm{T}}, \ldots, w_r^{\mathrm{T}}]^{\mathrm{T}}$ such that $w^{\mathrm{T}}[zI - A_e, B_e] = 0$ for $w \neq 0$. Then, $w_r^{\mathrm{T}} B = 0$ and

$$\begin{aligned}
w_0^{\mathrm{T}}(1+\tau z) - w_r^{\mathrm{T}} A_r \tau &= 0 \\
-w_0^{\mathrm{T}} + w_1^{\mathrm{T}}(1+\tau z) - w_r^{\mathrm{T}} A_{r-1} \tau &= 0 \\
&\vdots \\
w_{r-2}^{\mathrm{T}} + w_{r-1}^{\mathrm{T}}(1+\tau z) - w_r^{\mathrm{T}} A_1 \tau &= 0 \\
w_{r-1}^{\mathrm{T}} + w_r^{\mathrm{T}}(zI - A_0)\tau &= 0.
\end{aligned}$$

Multiplying subsequent equations by $(1+\tau z)^k$ for $k = 0, \ldots, r$ and summing them up, we get

$$w_r^{\mathrm{T}} \tau \left[(1+\tau z)^r (zI - A_0) - \sum_{k=1}^{r} (1+\tau z)^{r-k} A_k \right] = 0,$$

which is equivalent to $w_r^{\mathrm{T}} \left[zI - A_0 - \sum_{k=-r}^{-1} (1+\tau z)^k A_{-k} \right] = 0$. Moreover, each w_k for $k = 0, \ldots, r-1$ depends linearly on w_r, i.e. $w_k = \tau \sum_{i=1}^{k+1} (1+\tau z)^{-k} A_{r-k+i-1}^{\mathrm{T}} w_r$. Thus, $w \neq 0$ if and only if $w_r \neq 0$. Taking $v := w_r$ results in $v^{\mathrm{T}}[zI - A_0 - \sum_{k=-r}^{-1}(1+\tau z)^k A_{-k}, B] = 0$, which contradicts (21).

Finally, using the above considerations and the fact that by Lemma 2 stabilizabilities of (2) and (18) are equivalent, it follows that in case $\mathbb{T} = \tau\mathbb{Z}$ the stabilizability of (2) is equivalent to the stabilizability of (23). For $\mathbb{T} = \mathbb{R}$ this equivalence was shown in [9]. □

4. CONCLUSION

Although theories of differential and difference equations are inherently different, time scales based formalism allows an effective unification of the two coexisting paradigms. Furthermore, such formalism is expected to be a right tool for further extension, since time scales incorporate other cases (such as nonuniformly sampled systems, interval models, hybrid systems, etc.). In this paper, linear retarded time-delay systems, defined on a homogeneous time scale, are addressed. Necessary and sufficient conditions for exponential stability are formulated in terms of the characteristic equation. This approach is further used to present feedback stabilizability criteria. The future work will be devoted to extend obtained results to the case of more general time scales.

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APPENDIX A

Lemma 3 ([18])**.** *The condition* $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$ *holds if and only if for every* $\lambda \in \mathbb{C} : \operatorname{rank}[\lambda I - A, B] = n$.

Lemma 4 ([18])**.** *The condition* $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = r < n$ *holds if and only if there is an invertible matrix* T *such that the matrices* $\widetilde{A} = T^{-1}AT$ *and* $\widetilde{B} = T^{-1}B$ *take the forms*

$$\widetilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \ \widetilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{24}$$

where A_{11} *is* $r \times r$*,* B_1 *is* $r \times m$*, and the pair* (A_{11}, B_1) *is controllable.*

Corollary 1. *For some* $\lambda \in \mathbb{C}$, $\operatorname{rank}[\lambda I - A, B] < n$ *if and only if there is an invertible matrix* T *such that the matrices* $\widetilde{A} = T^{-1}AT$ *and* $\widetilde{B} = T^{-1}B$ *take the forms* (24)*, where* A_{11} *is* $r \times r$*,* B_1 *is* $r \times m$*, the pair* (A_{11}, B_1) *is controllable, and* λ *is an eigenvalue of* A_{22}*.*

Proof. First, observe that $\operatorname{rank}[\lambda I - A, B] = \operatorname{rank}[\lambda I - \widetilde{A}, \widetilde{B}]$ for $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$ and any invertible matrix T.

Sufficiency. Since $\operatorname{rank}[\lambda I_{n-r} - A_{22}] < n - r$, then $\operatorname{rank}[\lambda I - \widetilde{A}, \widetilde{B}] < n$, so also $\operatorname{rank}[\lambda I - A, B] < n$.

Necessity. From Lemmas 3 and 4 it follows that there is an invertible matrix T such that the matrices $\widetilde{A} = T^{-1}AT$ and $\widetilde{B} = T^{-1}B$ take forms (24), where A_{11} is $r \times r$, B_1 is $r \times m$, and the pair (A_{11}, B_1) is controllable. Observe that $\operatorname{rank}[\lambda I - \widetilde{A}, \widetilde{B}] = \operatorname{rank}[\lambda I_r - A_{11}, B_1] + \operatorname{rank}[\lambda I_{n-r} - A_{22}]$. By Lemmas 3 and 4, $\operatorname{rank}[\lambda I_r - A_{11}, B_1] = r$. Thus, $\operatorname{rank}[\lambda I_{n-r} - A_{22}] < n - r$, which means that λ is an eigenvalue of A_{22}. □

Lemma 5. *If* $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$*, then for every polynomial* $w(\lambda) = \lambda^n + w_{n-1}\lambda^{n-1} + \cdots + w_0$ *there is a matrix* K *such that* $\chi_{A+BK} = w$*.*

Proof of Theorem 6. Necessity. Suppose that for some $\lambda \in \mathbb{C}$, $\operatorname{rank}[\lambda I - A, B] < n$ and $\lambda \notin \mathscr{S}(\mathbb{T})$. From Corollary 1 there is an invertible matrix T such that the matrices $\widetilde{A} = T^{-1}AT$ and $\widetilde{B} = T^{-1}B$ take forms (24), where A_{11} is $r \times r$, B_1 is $r \times m$, the pair (A_{11}, B_1) is controllable, and λ is an eigenvalue of A_{22}. Let $K = (K_1, K_2)$ be an $m \times n$ matrix with K_1 being $m \times r$. Then, $\chi_{\widetilde{A}+\widetilde{B}K} = \chi_{A_{11}+B_1K_1}\chi_{A_{22}}$. Thus, the unstable eigenvalue λ cannot be changed by feedback, which means that the system is not stabilizable.

Sufficiency. Suppose that the system is not stabilizable. Then, from Lemma 5 it follows that $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = r < n$, and from Lemma 3, $\operatorname{rank}[\lambda I - A, B] < n$ for some $\lambda \in \mathbb{C}$. From Corollary 1, this λ must be an eigenvalue of A_{22}. Since the system is not stabilizable and only the eigenvalues of A_{22} cannot be changed by feedback, at least one of the eigenvalues of A_{22} must lie outside the stability set $\mathscr{S}(\mathbb{T})$. This contradicts the condition stated in the theorem. □

APPENDIX B

Exercise 2.51 from [4]**:** *Show that if* $\mathbb{T}$ *has constant graininess* $\tau \geq 0$ *and if* α *is constant with* $1+\alpha\tau \neq 0$*, then* $\mathrm{e}_\alpha(t+s,0) = \mathrm{e}_\alpha(t,0)\mathrm{e}_\alpha(s,0)$ *for all* $s,t \in \mathbb{T}$.

Solution. If $\tau > 0$ and $\mathbb{T} = \tau\mathbb{Z}$, then the exponential function can be written as $\mathrm{e}_\alpha(t,0) = (1+\alpha\tau)^{\frac{t}{h}}$ for all $t \in \mathbb{T}$. Therefore,

$$\mathrm{e}_\alpha(t+s,0) = (1+\alpha\tau)^{\frac{t+s}{h}} = (1+\alpha\tau)^{\frac{t}{h}}(1+\alpha\tau)^{\frac{s}{h}} = \mathrm{e}_\alpha(t,0)\mathrm{e}_\alpha(s,0)$$

for all $t,s \in \mathbb{T}$. If $\tau = 0$ and $\mathbb{T} = \mathbb{R}$, then the exponential function can be written as $\mathrm{e}_\alpha(t,0) = \mathrm{e}^{\alpha t}$ for all $t \in \mathbb{T}$. Therefore,

$$\mathrm{e}_\alpha(t+s,0) = \mathrm{e}^{\alpha(t+s)} = \mathrm{e}^{\alpha t}\mathrm{e}^{\alpha s} = \mathrm{e}_\alpha(t,0)\mathrm{e}_\alpha(s,0)$$

for all $t,s \in \mathbb{T}$.

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Lineaarse ajahilistumise süsteemi stabiilsus ja stabiliseerimine homogeensel ajaskaalal

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On välja pakutud tarvilikud ja piisavad tingimused eksponentsiaalse stabiilsuse jaoks lineaarse ajahilistumise süsteemis, defineerituna homogeensel skaalal. Tingimused on sõnastatud, kasutades karakteristlikku võrrandit, mis on süsteemiga seotud. Sellist lähenemist on hiljem kasutatud tagasiside stabiliseerimise kriteeriumi saamiseks.