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MATHEMATICS

On boundedness inequalities in the variation of certain Schurer-type operators

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Abstract. This paper is concerned with boundedness inequalities in the variation for the higher order derivatives of general Schurer-type operators. In particular, the boundedness inequalities in the variation for the higher order derivatives of the Bernstein–Schurer, Kantorovich–Schurer, and Durrmeyer–Schurer operators are derived.

Key words: approximation theory, Schurer-type operators, boundedness inequalities, variation detracting property, absolutely continuous functions.

1. INTRODUCTION

The Bernstein polynomials,

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \ \ (x \in [0,1]),$$

have influenced many branches of the approximation theory and their properties have been prototypes for our research as well.

Dealing with the class of functions of bounded variation $BV[0,1]$, the Bernstein polynomials have the total variation diminishing property

$$V_{[0,1]}[B_n f] \le V_{[0,1]}[f], \tag{1.1}$$

where $V_{[0,1]}[f]$ is the total variation of f and $f \in BV[0,1]$ (see [10], which is the first paper in this direction). In [2] this has been called the *variation detracting property* (VDP). The total variation diminishing property of this kind is known for many positive operators. In the case of the Kantorovich operators

$$(K_n f)(x) = (n+1)\sum_{k=0}^{n} p_{k,n}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du \quad (x \in [0,1]), \tag{1.2}$$

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where

$$p_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k},$$

the variation detracting property holds as follows.

Theorem A ([2], Proposition 3.3)**.** *If $f \in BV[0,1]$, then*

$$V_{[0,1]}[K_n f] \le V_{[0,1]}[f].$$

Also the Durrmeyer operators (see [7])

$$(D_n f)(x) = (n+1) \sum_{k=0}^{n} p_{k,n}(x) \int_0^1 p_{k,n}(t) f(t) dt \quad (x \in [0,1]), \tag{1.3}$$

have the variation detracting property.

For the Bernstein operator there has been some interest in investigating the variation detracting property for the derivatives in the form (see, e.g. [9])

$$V_{[0,1]}[(B_n f)'] \le V_{[0,1]}[f']. \tag{1.4}$$

In this paper we investigate boundedness inequalities for the higher order derivatives of some, quite general, Schurer-type operators, in particular cases the Bernstein–Schurer, the Kantorovich–Schurer, and the Durrmeyer–Schurer operators.

Let L_n be a polynomial positive operator, i.e., we have a polynomial $L_n f \ge 0$ on $[0,a]$ for every $f \ge 0$, $f \in BV[0,b]$, $(a, b > 0)$.

Due to (1.1) and (1.4) there arises the question of determining the constant $M_r > 0$, independent of $f, f^{(r)} \in BV[0,b]$, for which

$$V_{[0,a]}[(L_n f)^{(r)}] \le M_r V_{[0,b]}[f^{(r)}], \quad r = 0, 1, 2, \ldots. \tag{1.5}$$

We call inequality (1.5) the boundedness inequality. If $M_r \le 1$, we call inequality (1.5) the variation detracting property.

2. BOUNDEDNESS INEQUALITIES FOR DERIVATIVES OF SCHURER-TYPE OPERATORS

In [1,4,11,12], certain Schurer-type operators are defined and their approximation properties are investigated. We investigate boundedness inequalities (1.5) of some general Schurer-type operators in a unified approach. Let

$$(U_{n,p,a} f)(x) := \sum_{k=0}^{n+p} p_{k,n+p,a}(x) F_{k,n,p}(f), \; x \in [0,a], \tag{2.1}$$

where $F_{k,n,p}(f)$ is some positive linear functional of $f \in C[0, 1+p]$, $p = 0, 1, 2, \ldots$, and

$$p_{k,m,a}(x) := \frac{1}{a^m} \binom{m}{k} x^k (a-x)^{m-k}, \; k = 0, \ldots, m, \; x \in [0,a]. \tag{2.2}$$

We consider here the following cases:

1. If in (2.1) we put $a = 1$ and

$$F_{k,n,p}(f) := f\left(\frac{k}{n}\right), \tag{2.3}$$

then we get the Bernstein–Schurer operator $B_{n,p}$; the subcase $p = 0$ gives us the Bernstein operator B_n.

2. If in (2.1) we put $a = 1$ and

$$F_{k,n,p}(f) := (n+p+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt = \frac{n+p+1}{n+1}\int_0^1 f\left(\frac{k+v}{n+1}\right)dv, \tag{2.4}$$

then we get the Kantorovich–Schurer operator $K_{n,p}$; the subcase $p = 0$ gives us the Kantorovich operator K_n.

3. If in (2.1) we put $a = p+1$ and

$$F_{k,n,p}(f) := \frac{n+p+1}{p+1}\int_0^{p+1} p_{k,n+p,p+1}(t)f(t)dt, \tag{2.5}$$

then we get the Durrmeyer–Schurer operator $D_{n,p}$; the subcase $p = 0$ gives us the Durrmeyer operator D_n.

Since $U_{n,p,a}f$ as a polynomial is continuously differentiable on $[0,a]$, then for the left-hand side of (1.5) it is known that

$$V_{[0,a]}[(U_{n,p,a}f)^{(r)}] = \int_0^a |(U_{n,p,a}f)^{(r+1)}(x)|dx.$$

So let us find the $r+1$-th derivative of the polynomial $U_{n,p,a}f$.

We introduce differences by the first index:

$$\begin{aligned} \triangle^0 F_{k,n,p} &:= F_{k,n,p}, \quad \triangle^1 F_{k,n,p} \equiv \triangle F_{k,n,p} := F_{k+1,n,p} - F_{k,n,p}, \\ \triangle^r F_{k,n,p} &:= \triangle(\triangle^{r-1}F_{k,n,p}), \quad (r = 2,3,...). \end{aligned}$$

We use next a lemma, which is generalized from [14], Chap. II, §19, Lemma 2 (see also [6], p. 306, formula (2.3)).

Lemma 1. *For $r = 0,1,...,n+p,\ p = 0,1,...$ we have*

$$\frac{d^r}{dx^r}(U_{n,p,a}f)(x) = \frac{r!}{a^r}\binom{n+p}{r}\sum_{k=0}^{n+p-r} p_{k,n+p-r,a}(x)\triangle^r F_{k,n,p}(f). \tag{2.6}$$

This assertion can be proved by induction as in the book [14].

The following proposition gives us a general idea for studying the variation detracting property for operators (2.1).

Proposition 1. *Let $r = 0,1,...,n+p-1$. Then*

$$V_{[0,a]}[(U_{n,p,a}f)^{(r)}] \le \frac{r!}{a^r}\binom{n+p}{r}\sum_{k=0}^{n+p-r-1} |\triangle^{r+1}F_{k,n,p}(f)|. \tag{2.7}$$

Proof. Since the beta function yields

$$\int_0^a p_{k,m,a}(x)dx = \frac{a}{m+1}, \tag{2.8}$$

by Lemma 1 we have

$$\begin{aligned} V_{[0,a]}[(U_{n,p,a}f)^{(r)}] &= \int_0^a |(U_{n,p,a}f)^{(r+1)}(x)|dx \\ &\le \frac{a}{n+p-r}\frac{(r+1)!}{a^{r+1}}\binom{n+p}{r+1}\sum_{k=0}^{n+p-r-1}|\triangle^{r+1}F_{k,n,p}(f)| \\ &= \frac{r!}{a^r}\binom{n+p}{r}\sum_{k=0}^{n+p-r-1}|\triangle^{r+1}F_{k,n,p}(f)|. \end{aligned}$$

□

Next, we have to express differences through derivatives. First, we need a definition of $r-1$ times absolutely continuous functions on the interval $[0,b]$ (compare, e.g. [3], p. 7).

Definition 1. *We say that $f \in AC^{r-1}[0,b]$, $r \in \mathbb{N}$, the space of all $(r-1)$-times absolutely continuous functions on $[0,b]$, if $f(x)$ admits for every $x \in [0,b]$ the representation*

$$f(x) = \sum_{k=0}^{r-1} A_k x^k + \int_0^x du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{r-2}} du_{r-1} \int_0^{u_{r-1}} g(u_r) du_r$$

for some $g \in L^1[0,b]$ and some constants A_i, $i = 0,...,r-1$.

We introduce differences

$$\triangle_h^1 f(x) \equiv \triangle_h f(x) := f(x+h) - f(x),\ \triangle_h^k f(x) := \triangle_h(\triangle_h^{k-1} f(x))\ (k = 2,3,...).$$

The class $AC^{r-1}[0,b]$ allows us to represent the differences $\triangle_h^r f(x)$ via the derivatives $f^{(r)}$ of $f \in AC^{r-1}[0,b]$. The next lemma is from [13] (see Chap. 3, §3, formula (4)).

Lemma 2. *Let $f \in AC^{r-1}[0,p+1]$, where $p = 0,1,2,...$ and $r \in \mathbb{N}$. Moreover, let $n \in \mathbb{N}$, $0 \le k \le n+p-r-1$, $0 \le v \le 1$. Then*

$$\triangle_{1/n}^r f\left(\frac{k+v}{n}\right) = \int_{[0,1/n]^r} f^{(r)}\left(\frac{k+v}{n} + t_1 + ... + t_r\right) dt_1 ... dt_r. \tag{2.9}$$

Now we have to calculate, according to (2.7), the sum of differences (2.9).

Lemma 3. *Let $f \in AC^{r-1}[0,p+1]$, $r = 1,2,...,n+p-1$. Then for $0 \le v \le 1$ we have*

$$\sum_{k=0}^{m} \left|\triangle_{1/n}^r f\left(\frac{k+v}{n}\right)\right| \le \frac{1}{n^{r-1}} \int_0^{\frac{m+r}{n}} \left|f^{(r)}\left(u + \frac{v}{n}\right)\right| du, \tag{2.10}$$

where $m \le n+p-r$ for $v = 0$ and $m \le n+p-r-1$ for $0 < v \le 1$.

Proof. By Lemma 2 we have

$$\begin{aligned} \left|\triangle_{1/n}^r f\left(\frac{k+v}{n}\right)\right| &\le \int_0^{1/n} dt_1 ... \int_0^{1/n} dt_{r-1} \int_0^{1/n} \left|f^{(r)}\left(\frac{k+v}{n} + t_1 + ... + t_r\right)\right| dt_r \\ &= \int_0^{1/n} dt_1 ... \int_0^{1/n} dt_{r-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left|f^{(r)}\left(\frac{v}{n} + t_1 + ... + t_r\right)\right| dt_r. \end{aligned}$$

Taking the sum we get

$$\sum_{k=0}^{m} \left|\triangle_{1/n}^r f\left(\frac{k+v}{n}\right)\right| \le \int_0^{1/n} dt_1 ... \int_0^{1/n} dt_{r-1} \int_0^{\frac{m+1}{n}} \left|f^{(r)}\left(\frac{v}{n} + t_1 + ... + t_r\right)\right| dt_r. \tag{2.11}$$

By introducing the new variables $u_1 = t_1, ..., u_{r-1} = t_{r-1}, u_r = t_1 + ... + t_r$, we have

$$0 \le u_i \le \frac{1}{n}, \quad i = 1,...,r-1, \tag{2.12}$$

$$u_1 + ... + u_{r-1} \le u_r \le u_1 + ... + u_{r-1} + \frac{m+1}{n} \tag{2.13}$$

and the Jacobian determinant $J = 1$. We get from (2.12) and (2.13) the estimate

$$0 \leq u_r \leq \frac{m+r}{n}.$$

Hence, the integral on the right-hand side of inequality (2.11) is estimated by

$$\int_0^{1/n} du_1 ... \int_0^{1/n} du_{r-1} \int_0^{\frac{m+r}{n}} \left| f^{(r)} \left(\frac{v}{n} + u_r \right) \right| du_r = \frac{1}{n^{r-1}} \int_0^{\frac{m+r}{n}} \left| f^{(r)} \left(u + \frac{v}{n} \right) \right| du. \tag{2.14}$$

From (2.11) and (2.14) we obtain our assertion. □

Let us first investigate the boundedness inequality for the derivatives of the Bernstein–Schurer polynomials.

Theorem 1. *Let $f \in AC^r[0, p+1]$, $r = 0, 1, ..., n+p-1$. Then*

$$V_{[0,1]}[(B_{n,p}f)^{(r)}] \leq \frac{(n+p)!}{(n+p-r)!n^r} V_{[0,p+1]}[f^{(r)}]. \tag{2.15}$$

Proof. For the Bernstein–Schurer polynomials $F_{k,n,p}(f) = f(\frac{k}{n})$, $a = 1$. In the case $r = 0$ the proof is almost identical to the proof of Proposition 3.1 in [2]. In the case $r = 1, ..., n+p-1$ by Proposition 1 and Lemma 3 ($v = 0$) we have

$$\begin{aligned} V_{[0,1]}[(B_{n,p}f)^{(r)}] &\leq \frac{(n+p)!}{(n+p-r)!n^r} \int_0^{p+1} |f^{(r+1)}(u)| du \\ &= \frac{(n+p)!}{(n+p-r)!n^r} V_{[0,p+1]}[f^{(r)}]. \end{aligned}$$

□

As a corollary, we get now a statement for the Bernstein operators, proved also in [14], Chap. II, §19, Lemma 3.

Corollary 1. *Let $f \in AC^r[0,1]$, $r = 0, 1, ..., n-1$. Then for the arbitrary derivatives of the Bernstein operator the VDP holds, i.e.*

$$V_{[0,1]}[(B_n f)^{(r)}] \leq \frac{n!}{(n-r)!n^r} V_{[0,1]}[f^{(r)}]. \tag{2.16}$$

For the proof we take in Theorem 1 $p = 0$.

Similarly to the case of derivatives of the Bernstein–Schurer polynomials we can investigate the boundedness inequality for the derivatives of the Kantorovich–Schurer polynomials.

Theorem 2. *Let $f \in AC^r[0, p+1]$, $r = 0, 1, ..., n+p-1$. Then*

$$V_{[0,1]}[(K_{n,p}f)^{(r)}] \leq \frac{(n+p+1)!}{(n+p-r)!(n+1)^{r+1}} V_{[0,p+1]}[f^{(r)}]. \tag{2.17}$$

Proof. In the case $r = 0$ the proof is almost identical to the proof of Proposition 3.3 in [2]. In the case $r = 1, ..., n+p-1$ by Proposition 1 ($a = 1$) and Lemma 3 we have

$$\begin{aligned} V_{[0,1]}[(K_{n,p}f)^{(r)}] &\leq \frac{(n+p)!}{(n+p-r)!} \frac{n+p+1}{n+1} \sum_{k=0}^{n+p-r-1} \left| \triangle^{r+1}_{1/(n+1)} \int_0^1 f\left(\frac{k+v}{n+1}\right) dv \right| \\ &\leq \frac{(n+p+1)!}{(n+p-r)!} \frac{1}{n+1} \int_0^1 \sum_{k=0}^{n+p-r-1} \left| \triangle^{r+1}_{1/(n+1)} f\left(\frac{k+v}{n+1}\right) \right| dv \\ &\leq \frac{(n+p+1)!}{(n+p-r)!(n+1)^{r+1}} \int_0^1 dv \int_0^{\frac{n+p}{n+1}} |f^{(r+1)}(u + \frac{v}{n+1})| du. \end{aligned}$$

Taking $s = u + \frac{v}{n+1}$, $t = u$ we have

$$\begin{aligned} V_{[0,1]}[(K_{n,p}f)^{(r)}] &\leq \frac{(n+p+1)!}{(n+p-r)!(n+1)^{r+1}} \int_0^{p+1} |f^{(r+1)}(s)| ds \\ &= \frac{(n+p+1)!}{(n+p-r)!(n+1)^{r+1}} V_{[0,p+1]}[f^{(r)}]. \end{aligned}$$

□

We get now in the case $p = 0$

Corollary 2. *Let $f \in AC^r[0,1]$, $r = 0,1,...,n-1$. Then for the arbitrary derivatives of the Kantorovich operator the VDP holds, i.e.*

$$V_{[0,1]}[(K_n f)^{(r)}] \leq \frac{(n+1)!}{(n-r)!(n+1)^{r+1}} V_{[0,1]}[f^{(r)}].$$

To investigate the variation detracting property for the Durrmeyer–Schurer operators by Proposition 1 we need to calculate $\triangle^{r+1} F_{k,n,p}(f)$. By definition (2.5) we write

$$\triangle^{r+1} F_{k,n,p}(f) = \frac{n+p+1}{p+1} \int_0^{p+1} \triangle^{r+1} p_{k,n+p,p+1}(t) f(t) dt.$$

It appears that the differences of the basic polynomials $p_{k,n+p,p+1}(t)$ can be represented via derivatives. The next result in a particular case is obtained in [5], proof of Theorem II.6, p. 332; however, for the completeness of the presentation we will give an elementary proof.

Lemma 4. *For the basic polynomials in (2.2) the following equality*

$$\frac{d^r}{dx^r}\big(p_{k,m,a}(x)\big) = (-1)^r \frac{r!}{a^r} \binom{m}{r} \triangle^r p_{k-r,m-r,a}(x), \tag{2.18}$$

where $m \geq k \geq r \geq 0$, $x \in [0,a]$, holds.

Proof. We prove (2.18) by induction on r. For $r = 0$ it is obvious by definition. For $r = 1$ we get

$$\frac{d}{dx} p_{k,m,a}(x) = -\frac{m}{a}\big(p_{k,m-1,a}(x) - p_{k-1,m-1,a}(x)\big). \tag{2.19}$$

Let us assume that (2.18) holds for some $1 \leq r < m$. Differentiating (2.19) r times and using (2.18) we get

$$\begin{aligned} \frac{d^{r+1}}{dx^{r+1}}\big(p_{k,m,a}(x)\big) &= -\frac{m}{a}(-1)^r \frac{r!}{a^r} \binom{m-1}{r} \big(\triangle^r p_{k-r,m-r-1,a}(x) - \triangle^r p_{k-r-1,m-r-1,a}(x)\big) \\ &= (-1)^{r+1} \frac{(r+1)!}{a^{r+1}} \binom{m}{r+1} \triangle^{r+1} p_{k-r-1,m-r-1,a}(x). \end{aligned}$$

□

To estimate the constant M_r in (1.5) for the Durrmeyer–Schurer operators we need

Lemma 5. *Let*

$$P_{r,m,a}(t) := \sum_{k=0}^{m-r} p_{k+r,m+r,a}(t),\ m \geq r \geq 0,\ t \in [0,a].$$

Then

$$\max_{0 \leq t \leq a} P_{r,m,a}(t) = P_{r,m,a}\left(\frac{a}{2}\right) = \frac{1}{2^{m+r}} \sum_{k=0}^{m-r} \binom{m+r}{k+r}. \tag{2.20}$$

Proof. In the case $r=0$ for any $t\in[0,a]$ we have

$$P_{0,m,a}(t)=\sum_{k=0}^{m}p_{k,m,a}(t)=1.$$

Looking for the global maximum on $[0,a]$ we have $P_{r,m,a}(0)=P_{r,m,a}(a)=0,\ r\geq 1$. By Lemma 4 we have

$$\begin{aligned}\big(P_{r,m,a}(t)\big)' &= \sum_{k=0}^{m-r}p'_{k+r,m+r,a}(t)\\ &= -\frac{m+r}{a}\sum_{k=0}^{m-r}\big(p_{k+r,m+r-1,a}(t)-p_{k+r-1,m+r-1,a}(t)\big)\\ &= -\frac{m+r}{a}\big(p_{m,m+r-1,a}(t)-p_{r-1,m+r-1,a}(t)\big).\end{aligned}$$

So we get that the equation $P'_{r,m,a}(t)=0$, or the equation $p_{m,m+r-1,a}(t)=p_{r-1,m+r-1,a}(t)$, has its unique solution at $t=a/2$. □

Now we are in a position to get the variation detracting property for the derivatives of the Durrmeyer–Schurer operator. In what follows we generalize the known result of Derriennic [5] (see the proof of Proposition 4.1) for the Durrmeyer operators.

Theorem 3. *Let $f\in AC^r[0,p+1],\ r=0,1,...,n+p-1$. Then the relation*

$$\begin{aligned}V_{[0,p+1]}[(D_{n,p}f)^{(r)}] &\leq \frac{(n+p)!(n+p+1)!}{2^{n+p+r+1}(n+p-r)!(n+p+r+1)!}\\ &\quad\times\sum_{l=r+1}^{n+p}\binom{n+p+r+1}{l}V_{[0,p+1]}[f^{(r)}]\end{aligned}\tag{2.21}$$

holds.

Proof. From Proposition 1 we conclude

$$\|(D_{n,p}f)^{(r+1)}\|_1\leq\frac{(n+p)!}{(p+1)^r(n+p-r)!}\sum_{k=0}^{n+p-r-1}|\triangle^{r+1}F_{k,n,p}(f)|.\tag{2.22}$$

Since by definition (2.5)

$$\triangle^{r+1}F_{k,n,p}(f)=\frac{n+p+1}{p+1}\int_0^{p+1}\triangle^{r+1}p_{k,n+p,p+1}(t)f(t)dt,$$

from Lemma 4 we have

$$\triangle^{r+1}F_{k,n,p}(f)=\frac{(-1)^{r+1}(p+1)^r(n+p+1)!}{(n+p+r+1)!}\int_0^{p+1}p^{(r+1)}_{k+r+1,n+p+r+1,p+1}(t)f(t)dt.\tag{2.23}$$

By Leibniz formula for partial integration (see, e.g. [8], Chap. VIII, §1, p. 270, formula (5)) we get

$$\begin{aligned}\int_0^{p+1}p^{(r+1)}_{k+r+1,n+p+r+1,p+1}(t)f(t)dt &= \Big(f(t)p^{(r)}_{k+r+1,n+p+r+1,p+1}(t)-f'(t)p^{(r-1)}_{k+r+1,n+p+r+1,p+1}(t)\\ &\quad+\cdots+(-1)^rf^{(r)}(t)p_{k+r+1,n+p+r+1,p+1}(t)\Big)\Big|_0^{p+1}\\ &\quad+(-1)^{r+1}\int_0^{p+1}f^{(r+1)}(t)p_{k+r+1,n+p+r+1,p+1}(t)dt.\end{aligned}$$

Since $0 \le k \le n+p-r-1$, on the right-hand side all the terms in parentheses are zeros, and we have

$$\int_0^{p+1} p^{(r+1)}_{k+r+1,n+p+r+1,p+1}(t)f(t)dt = (-1)^{r+1}\int_0^{p+1} f^{(r+1)}(t)p_{k+r+1,n+p+r+1,p+1}(t)dt. \tag{2.24}$$

Now by (2.22), (2.23), and (2.24), using Lemma 5 we have Theorem 3 proved. □

Remark. It is interesting to notice that by Theorem 3 we get the VDP with the constant in inequality (2.21) less than 1.

Corollary 3. *Let $f \in AC^r[0,p+1]$, $r = 0,1,\dots,n+p-1$. Then the VDP for arbitrary derivatives of Durrmeyer–Schurer operators holds, i.e.*

$$V_{[0,p+1]}[(D_{n,p}f)^{(r)}] \le \frac{(n+p)!}{(n+p-r)!}\frac{(n+p+1)!}{(n+p+r+1)!}V_{[0,p+1]}[f^{(r)}].$$

In the case $p = 0$ we get the known result of Derriennic [5] (see the proof of Proposition 4.1) for the Durrmeyer operators.

Corollary 4. *Let $f \in AC^r[0,1]$, $r = 0,1,\dots,n-1$. Then the VDP for arbitrary derivatives of Durrmeyer operators holds, i.e.*

$$V_{[0,1]}[(D_nf)^{(r)}] \le \frac{n!}{(n-r)!}\frac{(n+1)!}{(n+r+1)!}V_{[0,1]}[f^{(r)}].$$

3. CONCLUSIONS

We investigated the boundedness inequalities for the higher order derivatives of some general Schurer-type operators in a unified approach. In particular, we proved the boundedness inequality for the higher order derivatives of the Bernstein–Schurer, Kantorovich–Schurer, and Durrmeyer–Schurer operators. Moreover, we proved the variation detracting property for the arbitrary derivatives of the Bernstein, Kantorovich, and Durrmeyer–Schurer operators. For the arbitrary derivatives of the Bernstein–Schurer and Kantorovich–Schurer operators only the boundedness inequality with the constant M_r on the right-hand side of the inequality that exceeds 1, holds.

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Teatud Schureri tüüpi operaatorite variatsiooni järgi tõkestatuse võrratused

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On uuritud teatud üldiste Schureri tüüpi operaatorite kõrgemat järku tuletiste variatsiooni järgi tõkestatuse võrratusi ühtse skeemi alusel. Sealhulgas on erijuhuna tõestatud Bernsteini, Kantorovichi ja Durrmeyeri-Schureri operaatorite mis tahes järku tuletiste variatsiooni mittekasvatamise omadus. Bernsteini-Schureri ja Kantorovichi-Schureri operaatorite mis tahes järku tuletiste korral on tõestatud variatsiooni järgi tõkestatuse võrratus.