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MATHEMATICS

Totally geodesic submanifolds of a trans-Sasakian manifold

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Abstract. We consider invariant submanifolds of a trans-Sasakian manifold and obtain the conditions under which the submanifolds are totally geodesic. We also study invariant submanifolds of a trans-Sasakian manifold satisfying $Z(X,Y).h = 0$, where Z is the concircular curvature tensor.

Key words: invariant submanifold, trans-Sasakian manifold, totally geodesic, semi-parallel, recurrent, pseudo-parallel, Ricci generalized pseudo-parallel.

1. INTRODUCTION

Invariant submanifolds of a contact manifold have been a major area of research for a long time since the concept was borrowed from complex geometry. It helps us to understand several important topics of applied mathematics; for example, in studying non-linear autonomous systems the idea of invariant submanifolds plays an important role [9]. A submanifold of a contact manifold is said to be totally geodesic if every geodesic in that submanifold is also geodesic in the ambient manifold. In 1985, Oubina [14] introduced a new class of almost contact manifolds, namely, trans-Sasakian manifold of type (α,β), which can be considered as a generalization of Sasakian, Kenmotsu, and cosymplectic manifolds. Trans-Sasakian structures of type $(0,0)$, $(0,\beta)$, and $(\alpha,0)$ are cosymplectic [2], β-Kenmotsu [10], and α-Sasakian [10], respectively. Kon [12] proved that invariant submanifolds of a Sasakian manifold are totally geodesic if the second fundamental form of the immersion is covariantly constant. On the other hand, any submanifold M of a Kenmotsu manifold is totally geodesic if and only if the second fundamental form of the immersion is covariantly constant, provided $\xi \in TM$ [11]. Recently, Sular and Özgür [16] proved some equivalent conditions regarding the submanifolds of a Kenmotsu manifold to be totally geodesic. Several studies ([5,17]) have been done on invariant submanifolds of trans-Sasakian manifolds. Recently, Sarkar and Sen [15] proved some equivalent conditions of an invariant submanifold of trans-Sasakian manifolds to be totally geodesic. In the present paper we rectify proofs of most of the major theorems of [15] and [17], show some theorems of [15] as corollary of our present results, and also introduce some new equivalent conditions for an invariant submanifold of a trans-Sasakian manifold to be totally geodesic.

2. PRELIMINARIES

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g), that is, ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a one-form, and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$ ([2,18]). The fundamental two-form Φ of the manifold is defined by

$$\Phi(X,Y) = g(X, \phi Y), \tag{2.4}$$

for $X, Y \in TM$.

An almost contact metric structure (ϕ, ξ, η, g) on a connected manifold M is called a trans-Sasakian structure [14] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [8], where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, fd/dt) = (\phi X - f\xi, \eta(X)d/dt),$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [3]

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for smooth functions α and β on M. Here we say that the trans-Sasakian structure is of type (α, β). From the formula (2.5) it follows that

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\bar{\nabla}_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

In a $(2n+1)$-dimensional trans-Sasakian manifold we also have the following:

$$\begin{aligned} S(X,\xi) &= 2n(\alpha^2 - \beta^2)\eta(X) - (2n-1)X\beta - \eta(X)\xi\beta - (\phi X)\alpha, && (2.8)\\ R(X,Y)\xi &= (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) && \\ &\quad -(X\alpha)\phi Y + (Y\alpha)\phi X - (X\beta)\phi^2 X + Y\beta\phi^2 X, && (2.9)\\ R(X,\xi)\xi &= (\alpha^2 - \beta^2)(X - \eta(X)\xi) + 2\alpha\beta\phi X + (\xi\alpha)\phi X + (\xi\beta)\phi^2 X, && (2.10) \end{aligned}$$

where S is the Ricci tensor of type $(0,2)$ and R is the curvature tensor of type $(1,3)$.

Let M be a submanifold of a contact manifold $\overline{M}$. We denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections of M and $\overline{M}$, respectively, and by $T^{\perp}(M)$ the normal bundle of M. Then for vector fields $X, Y \in TM$, the second fundamental form h is given by the formula

$$h(X,Y) = \bar{\nabla}_X Y - \nabla_X Y. \tag{2.11}$$

Furthermore, for $N \in T^{\perp}M$

$$A_N X = \nabla^{\perp}_X N - \bar{\nabla}_X N, \tag{2.12}$$

where $\nabla^{\perp}$ denotes the normal connection of M. The second fundamental form h and A_N are related by $g(h(X,Y),N) = g(A_N X, Y)$ [4].

The submanifold M is totally geodesic if and only if $h = 0$.

An immersion is said to be parallel and semi-parallel [6] if for all $X, Y \in TM$ we get $\nabla.h = 0$ and $R(X,Y).h = 0$, respectively.

It is said to be pseudo-parallel [7] if for all $X, Y \in TM$ we get

$$R(X,Y).h = fQ(g,h), \tag{2.13}$$

where f denotes a real function on M and $Q(E,T)$ is defined by

$$Q(E,T)(X,Y,Z,W) = -T((X \wedge_E Y)Z,W) - T(Z,(X \wedge_E Y)W), \tag{2.14}$$

where $X \wedge_E Y$ is defined by

$$(X \wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y.$$

If $f = 0$, the immersion is semi-parallel.

Similarly, an immersion is said to be 2-pseudo-parallel if for all $X,Y \in TM$ we get $R(X,Y).\nabla h = fQ(g,\nabla h)$, and Ricci generalized pseudo-parallel [13] if $R(X,Y).h = fQ(S,h)$, for all $X,Y \in TM$.

The second fundamental form h satisfying

$$(\nabla_Z h)(X,Y) = A(Z)h(X,Y), \tag{2.15}$$

where A is a nonzero one-form, is said to be recurrent. It is said to be 2-recurrent if h satisfies

$$(\nabla_X \nabla_Y h - \nabla_{\nabla_X Y} h)(Z,W) = B(X,Y)h(Z,W), \tag{2.16}$$

where B is a nonzero two-form.

Proposition 2.1. [5] *An invariant submanifold of a trans-Sasakian manifold is also trans-Sasakian.*

Proposition 2.2. [5] *Let M be an invariant submanifold of a trans-Sasakian manifold $\bar{M}$. Then we have*

$$\begin{aligned} h(X,\phi Y) &= \phi(h(X,Y)), && (2.17)\\ h(\phi X,\phi Y) &= -(h(X,Y)), && (2.18)\\ h(X,\xi) &= 0, && (2.19) \end{aligned}$$

for any vector fields X and Y on M.

For a Riemannian manifold, the concircular curvature tensor Z is defined by

$$Z(X,Y)V = R(X,Y)V - \frac{\tau}{n(n-1)}[g(Y,V)X - g(X,V)Y], \tag{2.20}$$

for vectors $X,Y,V \in TM$, where τ is the scalar curvature of M. We also have

$$(Z(X,Y).h)(U,V) = R^{\perp}(X,Y)h(U,V) - h(Z(X,Y)U,V) - h(U,Z(X,Y)V). \tag{2.21}$$

In the next section we consider the submanifold M to be tangent to ξ.

3. INVARIANT SUBMANIFOLDS OF A TRANS-SASAKIAN MANIFOLD WITH $\alpha,\beta =$ CONSTANT

Lemma 3.1. *If a non-flat Riemannian manifold has a recurrent second fundamental form, then it is semi-parallel.*

Proof. The second fundamental form h is said to be recurrent if

$$\nabla h = A \otimes h,$$

where A is an everywhere nonzero one-form.

We define a function e on M by

$$e^2 = g(h,h). \tag{3.1}$$

Then we have $e(Ye) = e^2A(Y)$. So we obtain $Ye = eA(Y)$, since f is nonzero. This implies that

$$X(Ye) - Y(Xe) = (XA(Y) - YA(X))e.$$

Therefore we get

$$[\bar{\nabla}_X\bar{\nabla}_Y - \bar{\nabla}_Y\bar{\nabla}_X - \bar{\nabla}_{[X,Y]}]e = [XA(Y) - YA(X) - A([X,Y])]e.$$

Since the left-hand side of the above equation is identically zero and e is nonzero on M by our assumption, we obtain

$$dA(X,Y) = 0, \tag{3.2}$$

that is, the one-form A is closed.

Now from $(\nabla_X h)(U,V) = A(X)h(U,V)$ we get

$$(\bar{\nabla}_U\bar{\nabla}_V h)(X,Y) - (\bar{\nabla}_{\bar{\nabla}_U V h})(X,Y) = [(\bar{\nabla}_U A)V + A(U)A(V)]h(X,Y) = 0.$$

Using (3.2) we get

$$(R(X,Y).h)(U,V) = [2dA(X,Y)]h(X,Y) = 0.$$

Therefore, for a recurrent second fundamental form we have

$$R(X,Y).h = 0$$

for any vectors X,Y on M.

If $e = 0$, then from (3.1) we get $h = 0$ and thus $R(X,Y).h = 0$.

Hence the lemma. □

Theorem 3.1. *An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is parallel.*

Proof. Since h is parallel, we have

$$(\nabla_X h)(Y,Z) = 0,$$

which implies

$$\nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) = 0.$$

Putting $Z = \xi$ in the above equation and applying (2.19) we obtain

$$h(Y,\nabla_X \xi) = 0. \tag{3.3}$$

So from (2.6) and the above equation (3.3) we obtain

$$\alpha h(X,Y) = \beta\varphi h(X,Y). \tag{3.4}$$

Applying φ to both sides of (3.4) we get

$$\alpha\varphi h(X,Y) = -\beta h(X,Y). \tag{3.5}$$

From (3.4) and (3.5) we conclude that

$$(\alpha^2 + \beta^2)h(X,Y) = 0.$$

Hence for a non-cosymplectic trans-Sasakian manifold $h(X,Y) = 0$, for all $X,Y \in TM$.

The converse part is trivial. Hence the result. □

Remark 3.1. In Theorem 3.1 [15] the authors proved the same result, but they actually proved $h(Y,\nabla_X\xi) = 0$, and $h(Y,\xi) = 0$, $\forall X,Y \in TM$. Since $\nabla_X\xi$ is not an arbitrary vector of TM, hence from this we can not conclude that the submanifold is totally geodesic.

Remark 3.2. Again in the proof of Theorem 4.8 [17] the authors assumed $\phi(h(X,Y)) = 0$, $\forall X,Y \in TM$, which is not true in general because this condition directly implies that the submanifold is totally geodesic.

Theorem 3.2. *An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is semi-parallel.*

Proof. Since h is semi-parallel, we have

$$(R(X,Y).h)(U,V) = 0, \tag{3.6}$$

which implies

$$R^{\perp}(X,Y)h(U,V) + h(R(X,Y)U,V) - h(U,R(X,Y)V) = 0. \tag{3.7}$$

Putting $V = \xi = Y$ and applying (2.19) we get from Eq. (3.7)

$$h(U,R(X,\xi)\xi) = 0.$$

So from (2.10) and (2.19) we get

$$(\alpha^2 - \beta^2)h(U,X) = 2\alpha\beta\varphi h(U,X). \tag{3.8}$$

Applying φ to both sides of Eq. (3.8) we obtain

$$(\alpha^2 - \beta^2)\varphi h(U,X) = -2\alpha\beta h(U,X). \tag{3.9}$$

So from (3.8) and (3.9) we conclude that

$$(\alpha^2 + \beta^2)^2 h(U,X) = 0.$$

Hence as in the previous case, for non-cosymplectic trans-Sasakian manifolds the invariant submanifold is totally geodesic. The converse part follows trivially. □

Now, by Lemma 3.1 we get that if a second fundamental form is recurrent, then it is semi-parallel. Also, the second fundamental form of a totally geodesic submanifold is trivially recurrent, so from Theorem 3.2 we obtain the following:

Corollary 3.1. *An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is recurrent.*

Remark 3.3. In Theorem 3.2 [15] the authors proved the above corollary, but they just showed that $h(Y,\nabla_X\xi) = 0$, and $h(Y,\xi) = 0$, $\forall X,Y \in TM$. Since $\nabla_X\xi$ is not an arbitrary vector of TM, we can not conclude from this that the submanifold is totally geodesic.

In [1] Aikawa and Matsuyama proved that if a tensor field T is 2-recurrent, then $R(X,Y).T = 0$. Also it can be easily seen that in a totally geodesic submanifold the second fundamental form is 2-recurrent. Therefore by Theorem 3.2 we also obtain the following:

Corollary 3.2. *An invariant submanifold of a non-cosymplectic trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is* 2*-recurrent.*

Remark 3.4. In Theorem 3.4 [15] the authors proved the above corollary, but they considered $\nabla_X\xi$ as an arbitrary vector of TM, and actually proved $h(Y,\nabla_X\xi) = 0, \forall X,Y \in TM$, hence the proof of Theorem 3.4 [15] is incorrect.

Theorem 3.3. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is* 2*-semi-parallel, provided* $\alpha^2(\alpha^2-3\beta^2)^2+\beta^2(\beta^2-3\alpha^2)^2\neq 0$.

Proof. Since, the second fundamental form is 2-semi-parallel, we have

$$(R(X,Y).(\nabla_U h))(Z,W)=0,$$

which implies

$$(R^{\perp}(X,Y)(\nabla_U h))(Z,W)-(\nabla_U h)(R(X,Y)Z,W)-(\nabla_U h)(Z,R(X,Y)W)=0.$$

Now,

$$(R^{\perp}(X,\xi)(\nabla_U h))(\xi,\xi)=0,$$

$$\begin{aligned}(\nabla_U h)(R(X,\xi)\xi,\xi)&=(\nabla_U h)((\alpha^2-\beta^2)(X-\eta(X)\xi)+2\alpha\beta\phi X,\xi)\\&=-h((\alpha^2-\beta^2)(X-\eta(X)\xi)+2\alpha\beta\phi X,-\alpha\phi U-\beta\phi^2U)\\&=\alpha(\alpha^2-\beta^2)h(X,\phi U)+2\alpha^2\beta h(\phi X,\phi U)+\beta(\alpha^2-\beta^2)h(X,\phi^2U)\\&\quad+2\alpha\beta^2h(\phi X,\phi^2U)\\&=\alpha(\alpha^2-3\beta^2)\phi h(X,U)+\beta(\beta^2-3\alpha^2)h(X,U).\end{aligned}$$

Similarly,

$$(\nabla_U h)(\xi,R(X,\xi)\xi)=\alpha(\alpha^2-3\beta^2)\phi h(X,U)+\beta(\beta^2-3\alpha^2)h(X,U). \tag{3.10}$$

So putting $Y=Z=W=\xi$ in (3.10) we obtain

$$\alpha(\alpha^2-3\beta^2)\phi h(X,U)+\beta(\beta^2-3\alpha^2)h(X,U)=0. \tag{3.11}$$

Applying ϕ on both sides of (3.11) we get

$$\alpha(\alpha^2-3\beta^2)h(X,U)=\beta(\beta^2-3\alpha^2)\phi h(X,U). \tag{3.12}$$

From (3.11) and (3.12) we conclude that

$$[\alpha^2(\alpha^2-3\beta^2)^2+\beta^2(\beta^2-3\alpha^2)^2]h(X,U)=0.$$

Hence the submanifold is totally geodesic. The converse holds trivially. □

Theorem 3.4. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is pseudo-parallel, provided* $[(\alpha^2-\beta^2-f)^2+4\alpha^2\beta^2]\neq 0$.

Proof. Since the second fundamental form is pseudo-parallel, we have

$$(R(X,Y).h)(U,V)=fQ(g,h)(X,Y,U,V),$$

which implies

$$\begin{aligned}&(R^{\perp}(X,Y))h(U,V)-h(R(X,Y)U,V)-h(U,R(X,Y)V)\\&\quad=f(-g(V,X)h(U,Y)+g(U,X)h(V,Y)-g(V,Y)h(U,X)+g(U,Y)h(V,X)).\end{aligned} \tag{3.13}$$

Putting $V=\xi=Y$ in Eq. (3.13) and applying (2.19) and (2.10) we obtain

$$-h(U,(\alpha^2-\beta^2)X+2\alpha\beta\varphi X)=f(-h(U,X)). \tag{3.14}$$

Applying φ to both sides of (3.14) we obtain

$$(\alpha^2-\beta^2-f)\varphi h(U,X)=2\alpha\beta h(U,X). \tag{3.15}$$

From (3.14) and (3.15) we conclude that

$$[(\alpha^2-\beta^2-f)^2+4\alpha^2\beta^2]h(U,X)=0.$$

Hence the submanifold is totally geodesic. The converse holds trivially. □

Theorem 3.5. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is* 2*-pseudo-parallel.*

Proof. Since, the second fundamental form is 2-pseudo-parallel, we have

$$(R(X,Y).\nabla_Z h)(U,V)=fQ(g,\nabla_Z h)(X,Y,U,V). \tag{3.16}$$

Now,

$$(R(X,Y).\nabla_Z h)(U,V)=R^{\perp}(X,Y)(\nabla_Z h)(U,V)-(\nabla_Z h)(R(X,Y)U,V)-(\nabla_Z h)(U,R(X,Y)V). \tag{3.17}$$

From (2.10) and (2.19) we have

$$(\nabla_Z h)(\xi,\xi)=0 \tag{3.18}$$

and

$$\begin{aligned}(\nabla_Z h)(R(X,\xi)\xi,\xi)&=-h(R(X,\xi)\xi,\nabla_Z\xi)\\&=\alpha(\alpha^2-\beta^2)h(X,\phi Z)+\beta(\alpha^2-\beta^2)h(X,\phi^2 Z)-2\alpha^2\beta h(\phi X,\phi Z)-2\alpha\beta^2 h(\phi X,\phi^2 Z)\\&=(\alpha^2+\beta^2)(\alpha\phi h(X,Z)+\beta h(X,Z)).\end{aligned} \tag{3.19}$$

So, putting $Y=U=V=\xi$ in (3.16) we obtain

$$2(\alpha^2+\beta^2)(\alpha\phi h(X,Z)+\beta h(X,Z))=0, \tag{3.20}$$

which implies

$$\alpha\phi h(X,Z)+\beta h(X,Z)=0. \tag{3.21}$$

Applying ϕ on both sides of Eq. (3.21) we get

$$\alpha h(X,Z)=\beta\phi h(X,Z). \tag{3.22}$$

Combining (3.21) and (3.22) we conclude that

$$[\alpha^2+\beta^2]h(X,Z)=0. \tag{3.23}$$

Hence the submanifold is totally geodesic. The converse holds trivially. □

Theorem 3.6. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if its second fundamental form is Ricci generalized pseudo-parallel, provided* $[(\alpha^2-\beta^2)^2(1-2nf)^2+4\alpha^2\beta^2]\neq 0$.

Proof. Since the submanifold is Ricci generalized pseudo-parallel, we have

$$(R(X,Y).h)(U,V) = fQ(S,h)(X,Y,U,V). \tag{3.24}$$

So,

$$\begin{aligned} &R(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) \\ &\quad = f(-S(V,X)h(U,Y) + S(U,X)h(V,Y) - S(V,Y)h(X,U) + S(U,Y)h(X,V)). \end{aligned} \tag{3.25}$$

Putting $Y = V = \xi$ and applying (2.19) we obtain

$$-h(U,R(X,\xi)\xi) = -fS(\xi,\xi)h(X,U).$$

Since α and β are constants, from (2.19), (2.10), and (2.8) we can write

$$(\alpha^2 - \beta^2)(1 - 2nf)h(X,U) = 2\alpha\beta\varphi h(X,U). \tag{3.26}$$

Applying φ on both sides of (3.26) we obtain

$$(\alpha^2 - \beta^2)(1 - 2nf)\varphi h(X,U) = -2\alpha\beta h(X,U). \tag{3.27}$$

From (3.26) and (3.27) we conclude that

$$[(\alpha^2 - \beta^2)^2(1 - 2nf)^2 + 4\alpha^2\beta^2]h(X,U) = 0.$$

Hence the submanifold is totally geodesic. The converse holds trivially. □

Theorem 3.7. *An invariant submanifold of a trans-Sasakian manifold is totally geodesic if and only if it satisfies $Z(X,Y).h = 0$, provided $(\alpha^2 - \beta^2 - \frac{\tau}{2n(2n+1)})^2 + 4\alpha^2\beta^2 \neq 0$.*

Proof. We have

$$(Z(X,Y).h)(U,V) = 0.$$

So from (2.21) we can write

$$R^{\perp}(X,Y)h(UV) - h(Z(X,Y)U,V) - h(Z(X,Y)U,V) = 0.$$

Putting $Y = V = \xi$ in the above equation and applying (2.19) we obtain

$$h(U,Z(X,\xi)\xi) = 0,$$

which implies that

$$h\left(U, (\alpha^2 - \beta^2)X + 2\alpha\beta\phi X - \frac{\tau}{2n(2n+1)}X\right) = 0, \text{since } h(X,\xi) = 0.$$

Simplifying we get

$$\left[(\alpha^2 - \beta^2) - \frac{\tau}{2n(2n+1)}\right] h(U,X) + 2\alpha\beta\phi h(U,X) = 0. \tag{3.28}$$

Applying ϕ on both sides of the above equation we get

$$\left[(\alpha^2 - \beta^2) - \frac{\tau}{2n(2n+1)}\right] \phi h(U,X) = 2\alpha\beta h(U,X). \tag{3.29}$$

From (3.28) and (3.29) we conclude

$$\left[\left(\alpha^2 - \beta^2 - \frac{\tau}{2n(2n+1)}\right)^2 + 4\alpha^2\beta^2\right] h(U,X) = 0.$$

The converse part follows trivially. Hence the result. □

4. CONCLUSION

A trans-Sasakian manifold can be regarded as a generalization of Sasakian, Kenmotsu, and cosymplectic structures. For an invariant submanifold of a trans-Sasakian manifold with constant coefficients the following conditions are equivalent under certain conditions:
• the submanifold is totally geodesic,
• the second fundamental form of the submanifold is parallel,
• the second fundamental form of the submanifold is semi-parallel,
• the second fundamental form of the submanifold is recurrent,
• the second fundamental form of the submanifold is 2-recurrent,
• the second fundamental form of the submanifold is 2-semi-parallel,
• the second fundamental form of the submanifold is pseudo-parallel,
• the second fundamental form of the submanifold is 2-pseudo-parallel,
• the second fundamental form of the submanifold is Ricci generalized pseudo-parallel,
• the second fundamental form of the submanifold satisfies $Z(X,Y).h = 0$.

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Täielikult geodeetilised trans-Sasaki muutkonna alammuutkonnad

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On vaadeldud invariantseid trans-Sasaki muutkonna alammuutkondi ja täiendatud nende täieliku geodeetilisuse tingimusi. Ühtlasi on uuritud trans-Sasaki muutkonna alammuutkondi, mille puhul $Z(X,Y).h = 0$, kus Z on kontsirkulaarne kõverustensor.