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MATHEMATICS

Strong summability methods in a Riesz-type family

Anna Šeletski* and Anne Tali

Department of Mathematics, Tallinn University, Narva mnt. 25, 10120 Tallinn, Estonia; atali@tlu.ee

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Abstract. We continue our studies on Riesz-type families of summability methods for functions and sequences, started in *Proc. Estonian Acad. Sci.*, 2008, **57**, 70–80 and *Math. Model. Anal.*, 2010, **15**, 103–112. Strong summability methods defined on the basis of a given Riesz-type family $\{A_\alpha\}$ are considered here. Inclusion theorems for these methods are proved. Our inclusion theorems allow us to compare the summability fields and speeds of convergence. The strong summability methods are also compared with ordinary summability methods A_α and with certain methods of statistical convergence. The proved theorems generalize different results that have already been published and are applied, in particular, to Riesz methods, generalized integral Nörlund methods, and Borel-type methods.

Key words: summability methods, integral methods, semi-continuous methods, strong summability methods, speed of convergence, statistical convergence, Riesz-type family, Riesz methods, generalized integral Nörlund methods, Borel-type methods.

1. INTRODUCTION AND PRELIMINARIES

Let $x = x(u)$ be the functions defined for $u \geq 0$, bounded and Lebesgue-measurable on every finite interval $[0, u_0]$. Denote the set of all these functions by X.

If the limit $\lim_{u\to\infty} x(u) = s$ exists, we say that $x = x(u)$ is convergent to s. Suppose that A is a transformation of functions $x = x(u)$ (or, in particular, of sequences $x = (x_n)$) into functions $Ax = y = y(u) \in X$. If the limit $\lim_{u\to\infty} y(u) = s$ exists, we say that $x = x(u)$ is convergent to s with respect to the summability method A (or x is summable to s by the method A) and write $x(u) \to s(A)$. If the function $y = y(u)$ is bounded, we say that x is bounded with respect to the method A, and write $x(u) = O(A)$. We denote by ωA the set of all these functions x, where the transformation A can be applied. The summability method A is said to be regular if for each $x \in X$

$$\lim_{u\to\infty} x(u) = s \Longrightarrow \lim_{u\to\infty} y(u) = s.$$

The most common summability method for functions x is an integral method A defined by the transformation

$$y(u) = \int_0^u a(u,v)x(v)dv, \tag{1.1}$$

where $a(u,v)$ is a certain function of two variables $(u \geq 0,\ v \geq 0)$ with $a(u,v) = 0$ for $v \geq u$. We also say that the integral method A is defined by the function $a(u,v)$.

* Corresponding author, annatar@hot.ee

The notion of a statistically convergent function is also used in this paper. According to [9] we say that x is statistically convergent to s and write $x(u) \to s(st)$ if for any $\varepsilon > 0$

$$\lim_{u\to\infty} \frac{|\mathscr{K}_{\varepsilon,u}|}{u} = 0,$$

where $|\mathscr{K}_{\varepsilon,u}|$ is the Lebesgue measure of the set

$$\mathscr{K}_{\varepsilon,u} = \{v \in [0,u] : |x(v) - s| \geq \varepsilon\}. \tag{1.2}$$

Generalizing the given notion of statistical convergence, we come to the following definition.

Definition 1. *Let A be a regular integral method defined by transformation* (1.1), *where $a(u,v)$ is some non-negative function. We say that $x = x(u)$ is A-statistically convergent to s and write $x(u) \to s(st_A)$ if for any $\varepsilon > 0$*

$$\lim_{u\to\infty} \int_{\mathscr{K}_{\varepsilon,u}} a(u,v)\, dv = 0,$$

where $\mathscr{K}_{\varepsilon,u}$ is the set defined by (1.2).

In particular, the notion of A-statistical convergence for the matrix case was first defined in [3] and later generalized and discussed in different papers (see [4] and [8] for references).

For converging sequences $x = (x_n)$ we focus on certain semi-continuous summability methods A defined by transformations

$$y(u) = \sum_{n=0}^{\infty} a_n(u)\, x_n \quad (u \geq 0),$$

where $a_n(u)$ $(n = 0, 1, 2, \ldots)$ are some functions from X.

One of the basic notions in this paper is the notion of the speed of convergence (see [10] and [12] and, in particular for sequences, [6] and [7]). Let $\lambda = \lambda(u)$ be a positive function from X such that $\lambda(u) \to \infty$ as $u \to \infty$. We say that a function $x = x(u)$ is convergent to s with speed λ if the finite limit

$$\lim_{u\to\infty} \lambda(u)\,[x(u) - s]$$

exists. Note that the limit can be zero. If we have $\lambda(u)\,[x(u) - s] = O(1)$ as $u \to \infty$, then x is said to be bounded with speed λ. We say that x is convergent or bounded with speed λ with respect to the summability method A if $Ax = y(u)$ is convergent or bounded with speed λ, respectively.

In our paper we study Riesz-type families of summability methods defined in [10] and [13].

Let $\{A_\alpha\}$ be a family of summability methods A_α, where[1] $\alpha \underset{(-)}{>} \alpha_0$ and which are defined by transformations of functions $x = x(u) \in \omega A_\alpha \subset X$ into functions $A_\alpha x = y_\alpha = y_\alpha(u) \in X$. Suppose that for any $\beta > \gamma \underset{(-)}{>} \alpha_0$ we have the relation

$$\omega A_\gamma \subset \omega A_\beta. \tag{1.3}$$

Definition 2. *A family $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ is said to be a Riesz-type family if for every $\beta > \gamma \underset{(-)}{>} \alpha_0$ relation* (1.3) *holds and the methods A_γ and A_β are connected by the relation*

$$y_\beta(u) = \frac{M_{\gamma,\beta}}{r_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1}\, r_\gamma(v)\, y_\gamma(v)\, dv \qquad (u > 0) \tag{1.4}$$

with

$$r_\beta(u) = M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1}\, r_\gamma(v)\, dv \qquad (u > 0), \tag{1.5}$$

[1] The notation $\alpha \underset{(-)}{>} \alpha_0$ means that we consider parameter values $\alpha > \alpha_0$ or $\alpha \geq \alpha_0$ with some fixed number α_0, depending on the specific situation.

where $r_\gamma = r_\gamma(u)$ and $r_\beta = r_\beta(u)$ are some positive functions from X and $M_{\gamma,\beta}$ is a constant depending on γ and β.

In other words, a Riesz-type family is a family where every two methods are connected through the connection formula

$$A_\beta = C_{\gamma,\beta} \circ A_\gamma \quad (\beta > \gamma \underset{(-)}{>} \alpha_0),$$

where $C_{\gamma,\beta}$ is the integral method defined by the function

$$c_{\gamma,\beta}(u,v) = \begin{cases} M_{\gamma,\beta}(u-v)^{\beta-\gamma-1} r_\gamma(v) / r_\beta(u) & \text{if } 0 \le v < u, \\ 0 & \text{if } v \ge u. \end{cases} \tag{1.6}$$

Next we introduce some examples of Riesz-type families (see, e.g., [15]).

Example 1. Consider the generalized Nörlund methods $A_\alpha = (N, u^{\alpha-1}, q(u))$, where $\alpha > 0$ and $q = q(u)$ is a positive function from X. These methods are defined by the transformation of x into $A_\alpha x = y_\alpha = y_\alpha(u)$ with

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \int_0^u (u-v)^{\alpha-1} q(v) x(v)\, dv \qquad (u > 0),$$

where $r_\alpha = r_\alpha(u) = \int_0^u (u-v)^{\alpha-1} q(v)\, dv$. These methods form a Riesz-type family because relations (1.4) and (1.5) are satisfied here for any $\beta > \gamma > 0$ with

$$M_{\gamma,\beta} = \frac{\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta-\gamma)}, \tag{1.7}$$

where $\Gamma(\cdot)$ is the Gamma-function. In particular, if $q(u) = 1$ $(u \ge 0)$, we have $r_\alpha(u) = u^\alpha/\alpha$ and the methods $(N, u^{\alpha-1}, q(u))$ turn into Riesz methods (R, α) (see [5]). The methods (R, α) form the Riesz-type family for $\alpha \ge 0$ if we take $y_0(u) = x(u)$ and $r_0(u) = 1$ for any $u \ge 0$.

Example 2. Let $\{A_\alpha\}$ be the family of generalized Nörlund methods $(N, p_\alpha(u), q(u))$ $(\alpha > \alpha_0)$, defined by the transformation

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \int_0^u p_\alpha(u-v) q(v) x(v)\, dv \qquad (u > 0)$$

with the help of positive functions $p = p(u) \in X$ and $q = q(u) \in X$ and the number α_0 such that $r_\alpha(u) = \int_0^u p_\alpha(u-v) q(v)\, dv > 0$, where $p_\alpha(u) = \int_0^u (u-v)^{\alpha-1} p(v)\, dv$. Here (1.4), together with (1.5) and (1.7), hold for any $\beta > \gamma > \alpha_0$. Hence $\{A_\alpha\}$ is a Riesz-type family.

Example 3. Consider the family $\{A_\alpha\}$ of Borel-type methods $A_\alpha = (B, \alpha, q_n)$. Let (q_n) be a non-negative sequence such that the power series $\sum q_n u^n$ has the radius of convergence $R = \infty$ and $q_n > 0$ at least for one $n \in \mathbb{N}$. Denote

$$r_\alpha(u) = \sum_{n=1}^\infty \frac{n!\, q_n u^{n+\alpha-1}}{\Gamma(n+\alpha)}$$

and define the methods (B, α, q_n) $(\alpha > -1/2)$ for converging sequences $x = (x_n)$ with the help of the transformation

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \sum_{n=1}^\infty \frac{n!\, q_n u^{n+\alpha-1}}{\Gamma(n+\alpha)} x_n \qquad (u > 0).$$

The methods (B, α, q_n) satisfy (1.4) and (1.5) with $M_{\gamma,\beta} = 1/\Gamma(\beta-\gamma)$ (see [13]). Thus $\{A_\alpha\}$ is a Riesz-type family. In particular, if $q_n = 1/n!$, we get the Borel-type methods $(B, \alpha) = (B, \alpha, 1/n!)$ (see [1,2]) which include the Borel method $B = (B, 1)$ (see [5]).

We need the following proposition (see [10] and [14]).

Proposition 1. *The methods $C_{\gamma,\beta}$ defined by* (1.6) *and* (1.5) *are regular for all $\beta > \gamma > \alpha_0$. These mehods are regular also for all $\beta > \gamma = \alpha_0$, provided that*

$$\lim_{u\to\infty}\int_0^u r_{\alpha_0}(v)\,dv = \infty. \tag{1.8}$$

In the present paper the authors continue their studies on Riesz-type families started in [14] and [15]. The main idea of the paper is to define the family of strong summability methods $[A_{\alpha+1}]_k$ on the basis of a given Riesz-type family $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ and to describe it with the help of different inclusion theorems. These strong summability methods are compared with each other by summability fields, i.e., by sets of functions x they converge, and by the speed of convergence. The strong summability methods are also compared with methods A_α and $A_{\alpha+1}$ and with certain methods of statistical convergence.

2. INCLUSION THEOREMS FOR STRONG SUMMABILITY METHODS $[A_{\alpha+1}]_k$

We start this section with the definition of strong summability methods for converging functions x, supposing that $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ is a Riesz-type family and $k = k(u)$ is a positive function from X.

Let us denote

$$\sigma^k_{\alpha+1}(u) = \frac{1}{r_{\alpha+1}(u)}\int_0^u r_\alpha(v)\,|y_\alpha(v)-s|^{k(v)}\,dv, \tag{2.1}$$

where $y_\alpha(u) = A_\alpha x$ $(x \in \omega A_\alpha)$ and $r_\alpha(u)$ and $r_{\alpha+1}(u)$ are defined by the family $\{A_\alpha\}$.

Definition 3. *Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ be a Riesz-type family and $k = k(u)$ be a positive function from X. We say that a function $x = x(u)$ is strongly convergent to s with respect to the method $A_{\alpha+1}$ (in short, $[A_{\alpha+1}]_k$-convergent) and write $x(u) \to s\,[A_{\alpha+1}]_k$, if $\sigma^k_{\alpha+1}(u) \to 0$ as $u \to \infty$.*

We say that a function $x = x(u)$ is strongly bounded with respect to the method $A_{\alpha+1}$ (in short, $[A_{\alpha+1}]_k$-bounded) and write $x(u) = O\,[A_{\alpha+1}]_k$ if

$$\frac{1}{r_{\alpha+1}(u)}\int_0^u r_\alpha(v)\,|y_\alpha(v)|^{k(v)}\,dv = O(1). \tag{2.2}$$

Thus we have defined the methods $[A_{\alpha+1}]_k$ $(\alpha \underset{(-)}{>} \alpha_0)$. In particular, if $A_\alpha = (N, u^{\alpha-1}, q(u))$, $A_\alpha = (N, p_\alpha(u), q(u))$ or $A_\alpha = (B, \alpha, q_n)$, then $r_\alpha(u)$ and $r_{\alpha+1}(u)$ were defined in Examples 1, 2 or 3, respectively.

We begin proving some inclusion theorems.

Theorem 1. *Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ be a Riesz-type family. Let $k = k(u)$ and $k' = k'(u)$ be two functions from X. Then the following statements are true for functions $x = x(u)$ and numbers s and $\beta > \gamma \underset{(-)}{>} \alpha_0$:*

(i) *if $x(u) \to s[A_{\gamma+1}]_k$, then $x(u) \to s[A_{\gamma+1}]_{k'}$ and if $x(u) = O[A_{\gamma+1}]_k$, then $x(u) = O[A_{\gamma+1}]_{k'}$, provided that $0 < k'(u) \le k(u) \le Mk'(u)$, where M is some positive constant*;

(ii) *if $x(u) \to s[A_{\gamma+1}]_k$, then $x(u) \to s\left(A_{\gamma+1}\right)$ and if $x(u) = O[A_{\gamma+1}]_k$, then $x(u) = O\left(A_{\gamma+1}\right)$, provided that $1 \le k(u) \le \sup_u k(u) = M < \infty$;*

(iii) *if $x(u) \to s[A_{\gamma+1}]_k$, then $x(u) \to s[A_{\beta+1}]_k$ and if $x(u) = O[A_{\gamma+1}]_k$, then $x(u) = O[A_{\beta+1}]_k$, provided that $k = k(u)$ is nonincreasing and $k(u) \ge 1$.*

Proof. Take w.l.o.g. $s = 0$.

(i) The quantity $\sigma^{k'}_{\gamma+1}(u)$ can be written in the form

$$\sigma^{k'}_{\gamma+1}(u) = \frac{1}{r_{\gamma+1}(u)}\int_0^u r_\gamma(v)\,\left|y_\gamma(v)\right|^{k'(v)}\,dv = \frac{1}{r_{\gamma+1}(u)}\int_0^u r_\gamma(v)\,\left[\left|y_\gamma(v)\right|^{k(v)}\right]^{\frac{k'(v)}{k(v)}}\,dv.$$

Notice that

$$1 \le \frac{k(u)}{k'(u)} \le M.$$

Denote

$$u_\gamma(v) = \begin{cases} \left|y_\gamma(v)\right|^{k(v)} & \text{if } \left|y_\gamma(v)\right| \ge 1, \\ 0 & \text{if } \left|y_\gamma(v)\right| < 1 \end{cases}$$

and

$$w_\gamma(v) = \begin{cases} \left|y_\gamma(v)\right|^{k(v)} & \text{if } \left|y_\gamma(v)\right| < 1, \\ 0 & \text{if } \left|y_\gamma(v)\right| \ge 1\,. \end{cases}$$

Thus we have the relations

$$\left|y_\gamma(v)\right|^{k(v)} = u_\gamma(v) + w_\gamma(v) \quad (v \ge 0),$$

$$\left|y_\gamma(v)\right|^{k'(v)} = \left[\left|y_\gamma(v)\right|^{k(v)}\right]^{\frac{k'(v)}{k(v)}} = \left[u_\gamma(v)\right]^{\frac{k'(v)}{k(v)}} + \left[w_\gamma(v)\right]^{\frac{k'(v)}{k(v)}},$$

$$\left[u_\gamma(v)\right]^{\frac{k'(v)}{k(v)}} \le u_\gamma(v) \le \left|y_\gamma(v)\right|^{k(v)},$$

$$\left[w_\gamma(v)\right]^{\frac{k'(v)}{k(v)}} \ge w_\gamma(v),\ \left[w_\gamma(v)\right]^{\frac{k'(v)}{k(v)}} \le \left[w_\gamma(v)\right]^{\frac{1}{M}}.$$

Using these relations, we get with the help of the Hölder inequality and (1.5)

$$\begin{aligned}
\sigma^{k'}_{\gamma+1}(u) &= \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left[\left|y_\gamma(v)\right|^{k(v)}\right]^{\frac{k'(v)}{k(v)}} dv \\
&= \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left[u_\gamma(v)\right]^{\frac{k'(v)}{k(v)}} dv + \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left[w_\gamma(v)\right]^{\frac{k'(v)}{k(v)}} dv \\
&\le \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v)\, u_\gamma(v) dv + \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left[w_\gamma(v)\right]^{\frac{1}{M}} dv \\
&\le \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_\gamma(v)\right|^{k(v)} dv + \left[\frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) dv\right]^{1-\frac{1}{M}} \\
&\quad \times \left[\frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) w_\gamma(v) dv\right]^{\frac{1}{M}} = O(1) \left\{ \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_\gamma(v)\right|^{k(v)} dv \right. \\
&\quad \left. + \left[\frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_\gamma(v)\right|^{k(v)} dv\right]^{\frac{1}{M}} \right\}.
\end{aligned}$$

Thus we have got the relation

$$\sigma^{k'}_{\gamma+1}(u) = O(1)\{\sigma^{k}_{\gamma+1}(u) + [\sigma^{k}_{\gamma+1}(u)]^{\frac{1}{M}}\}, \tag{2.3}$$

which implies our statement (i).

(ii) As

$$\left|y_{\gamma+1}(u)\right| \le \frac{M_{\gamma,\gamma+1}}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_\gamma(v)\right| dv, \tag{2.4}$$

using statement (i), we get

$$x(u) \to s[A_{\gamma+1}]_k \Longrightarrow x(u) \to s[A_{\gamma+1}]_1 \Longrightarrow x(u) \to s(A_{\gamma+1}).$$

The part of statement (ii) about boundednes follows from (2.4) in an analogous way.

(iii) According to (2.1) we have

$$\sigma^k_{\beta+1}(u) = \frac{1}{r_{\beta+1}(u)} \int_0^u r_\beta(v) \left|y_\beta(v)\right|^{k(v)} dv.$$

Now we get with the help of (1.4) and the Hölder inequality

$$\begin{aligned}
\sigma^k_{\beta+1}(u) &= \frac{1}{r_{\beta+1}(u)} \int_0^u r_\beta(v) \left| \frac{M_{\gamma,\beta}}{r_\beta(v)} \int_0^v (v-t)^{\beta-\gamma-1} r_\gamma(t) y_\gamma(t) dt \right|^{k(v)} dv \\
&= O(1) \frac{1}{r_{\beta+1}(u)} \int_0^u r_\beta(v) \left(\frac{1}{r_\beta(v)} \int_0^v (v-t)^{\beta-\gamma-1} r_\gamma(t) \left|y_\gamma(t)\right|^{k(v)} dt \right) \\
&\quad \times \left(\frac{1}{r_\beta(v)} \int_0^v (v-t)^{\beta-\gamma-1} r_\gamma(t) dt \right)^{k(v)-1} dv \\
&= O(1) \frac{1}{r_{\beta+1}(u)} \int_0^u \left(\int_0^v (v-t)^{\beta-\gamma-1} r_\gamma(t) \left|y_\gamma(t)\right|^{k(v)} dt \right) dv \\
&= O(1) \frac{1}{r_{\beta+1}(u)} \int_0^u \left(\int_0^v t^{\beta-\gamma-1} r_\gamma(v-t) \left|y_\gamma(v-t)\right|^{k(v)} dt \right) dv \\
&= O(1) \frac{1}{r_{\beta+1}(u)} \int_0^u t^{\beta-\gamma-1} \left(\int_t^u r_\gamma(v-t) \left|y_\gamma(v-t)\right|^{k(v)} dv \right) dt.
\end{aligned}$$

Thus we have shown that

$$\sigma^k_{\beta+1}(u) = O(1) \frac{1}{r_{\beta+1}(u)} \int_0^u t^{\beta-\gamma-1} \left(\int_0^{u-t} r_\gamma(v) \left|y_\gamma(v)\right|^{k(t+v)} dv \right) dt. \tag{2.5}$$

Denoting

$$u_\gamma(v,t) = \begin{cases} \left|y_\gamma(v)\right|^{k(t+v)} & \text{if } \left|y_\gamma(v)\right| \geq 1, \\ 0 & \text{if } \left|y_\gamma(v)\right| < 1 \end{cases}$$

and

$$w_\gamma(v,t) = \begin{cases} \left|y_\gamma(v)\right|^{k(t+v)} & \text{if } \left|y_\gamma(v)\right| < 1, \\ 0 & \text{if } \left|y_\gamma(v)\right| \geq 1, \end{cases}$$

we get the relations

$$\left|y_\gamma(v)\right|^{k(t+v)} = u_\gamma(v,t) + w_\gamma(v,t) \quad (v \geq 0,\ t \geq 0),$$

$$u_\gamma(v,t) \leq \left|y_\gamma(v)\right|^{k(v)},\ w_\gamma(v,t) \leq \left|y_\gamma(v)\right|.$$

Further we use also the notation

$$\sigma^1_{\gamma+1}(u) = \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_\gamma(v)\right| dv. \tag{2.6}$$

Developing (2.5) with the help of the last relations, we get

$$\begin{aligned}
\sigma^k_{\beta+1}(u) &= O(1)\left[\frac{1}{r_{\beta+1}(u)}\int_0^u t^{\beta-\gamma-1}\left(\int_0^{u-t} r_\gamma(v)\, u_\gamma(v,t)\, dv\right) dt\right.\\
&\quad \left. + \frac{1}{r_{\beta+1}(u)}\int_0^u t^{\beta-\gamma-1}\left(\int_0^{u-t} r_\gamma(v)\, w_\gamma(v,t)\, dv\right) dt\right]\\
&= O(1)\left[\frac{1}{r_{\beta+1}(u)}\int_0^u t^{\beta-\gamma-1}\left(\int_0^{u-t} r_\gamma(v)\left|y_\gamma(v)\right|^{k(v)} dv\right) dt\right.\\
&\quad \left. + \frac{1}{r_{\beta+1}(u)}\int_0^u t^{\beta-\gamma-1}\left(\int_0^{u-t} r_\gamma(v)\left|y_\gamma(v)\right| dv\right) dt\right]\\
&= O(1)\left[\frac{1}{r_{\beta+1}(u)}\int_0^u t^{\beta-\gamma-1} r_{\gamma+1}(u-t)\, \sigma^k_{\gamma+1}(u-t)\, dt\right.\\
&\quad \left. + \frac{1}{r_{\beta+1}(u)}\int_0^u t^{\beta-\gamma-1} r_{\gamma+1}(u-t)\, \sigma^1_{\gamma+1}(u-t)\, dt\right].
\end{aligned}$$

So we have proved the relation

$$\begin{aligned}
\sigma^k_{\beta+1}(u) &= O(1)\left[\frac{1}{r_{\beta+1}(u)}\int_0^u (u-t)^{\beta-\gamma-1} r_{\gamma+1}(t)\, \sigma^k_{\gamma+1}(t)\, dt\right.\\
&\quad \left. + \frac{1}{r_{\beta+1}(u)}\int_0^u (u-t)^{\beta-\gamma-1} r_{\gamma+1}(t)\, \sigma^1_{\gamma+1}(t)\, dt\right]. \qquad (2.7)
\end{aligned}$$

It follows from (2.7) that if $\sigma^k_{\gamma+1}(u) \to 0$, then $\sigma^k_{\beta+1}(u) \to 0$ (as $u \to \infty$), because $\sigma^k_{\gamma+1}(u) \to 0$ implies $\sigma^1_{\gamma+1}(u) \to 0$ by statement (i) and $C_{\gamma+1,\beta+1}$ is a regular method by Proposition 1. The part of statement (iii) about boundedness follows from (2.7) analogously. □

Remark 1.

(i) If we weaken the restrictions on k and k' allowing also the case $\frac{k(u)}{k'(u)} \to \infty$, then statement (i) of Theorem 1 is not true in general (see Remark 2 in [11]).

(ii) If $0 < k(u) < 1$, then statement (ii) is not true in general (see Remark 4 in [11]).

(iii) In particular, if $k(u) \equiv r$, then relation (2.5) takes the form

$$\sigma^r_{\beta+1}(u) = O(1)\left[\frac{1}{r_{\beta+1}(u)}\int_0^u (u-v)^{\beta-\gamma-1} r_{\gamma+1}(v)\, \sigma^r_{\gamma+1}(v)\, dv\right] \qquad (2.8)$$

and completes the proof of statement (iii) of Theorem 1.

Theorem 2. *Let* $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ *be a Riesz-type family. Then the following statements are true for functions* $x = x(u)$ *and numbers* s *and* $\gamma \geq \alpha_0$ *:*

(i) *if* $x(u) \to s\left(A_\gamma\right)$ *, then* $x(u) \to s[A_{\gamma+1}]_k$*, provided that* $\inf_u k(u) = m > 0$*, and provided that also condition* (1.8) *holds if* α_0 *is included*;

(ii) *if* $x(u) = O\left(A_\gamma\right)$ *, then* $x(u) = O[A_{\gamma+1}]_k$*, provided that* $\sup_u k(u) = M < \infty$.

Proof. Take w.l.o.g. $s = 0$.

(i) As $k(u) \geq m > 0$ for any $u \geq 0$ and $\left|y_\gamma(u)\right| \leq 1$ for sufficiently large u, we have $\left|y_\gamma(u)\right|^{k(u)} \leq \left|y_\gamma(u)\right|^m$ for sufficiently large u and therefore $\left|y_\gamma(u)\right|^{k(u)} \to 0$ if $y_\gamma(u) \to 0$. Thus $\sigma^k_{\gamma+1}(u) \to 0$ as $y_\gamma(u) \to 0$ (as $u \to \infty$) due to (2.1) and regularity of $C_{\gamma,\gamma+1}$.

Statement (ii) is also true because $y_\gamma(u) = O(1)$ implies $\left|y_\gamma(u)\right|^{k(u)} = O(1)$. □

Theorem 3. *Let* $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ *be a Riesz-type family. Suppose that* $1 \le k(u) \le \sup_u k(u) = M < \infty$. *Then the following statements are true for functions* $x = x(u)$ *and numbers* s *and* $\gamma \underset{(-)}{>} \alpha_0$:

(i) $x(u) \to s[A_{\gamma+1}]_k$ *if and only if* $x(u) \to s(A_{\gamma+1})$ *and* $\left|y_{\gamma+1}(u) - y_\gamma(u)\right|^{k(u)} \to 0(C_{\gamma,\gamma+1})$, *provided that* (1.8) *holds if* α_0 *is included*;

(ii) $x(u) = O[A_{\gamma+1}]_k$ *if and only if* $x(u) = O(A_{\gamma+1})$ *and* $\left|y_{\gamma+1}(u) - y_\gamma(u)\right|^{k(u)} = O(C_{\gamma,\gamma+1})$, *where the method* $C_{\gamma,\gamma+1}$ *is defined by* (1.6).

Proof. (i) *Necessity.* Denote[2]

$$\delta^k_{\gamma+1}(u) = \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_{\gamma+1}(v) - y_\gamma(v)\right|^{k(v)} dv. \tag{2.9}$$

Using the Minkowski inequality and the inequality

$$|a+b|^c \le |a|^c + |b|^c \quad (c \le 1) \tag{2.10}$$

with $c = k(u)/M$, we get

$$\begin{aligned}
[\delta^k_{\gamma+1}(u)]^{\frac{1}{M}} &= \left\{ \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left(\left|y_{\gamma+1}(v) - s + s - y_\gamma(v)\right|^{\frac{k(v)}{M}} \right)^M dv \right\}^{\frac{1}{M}} \\
&\le \left\{ \frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left(\left|y_{\gamma+1}(v) - s\right|^{\frac{k(v)}{M}} + \left|y_\gamma(v) - s\right|^{\frac{k(v)}{M}} \right)^M dv \right\}^{\frac{1}{M}} \\
&\le \left[\int_0^u \frac{r_\gamma(v)}{r_{\gamma+1}(u)} \left|y_{\gamma+1}(v) - s\right|^{k(v)} dv \right]^{\frac{1}{M}} + \left[\int_0^u \frac{r_\gamma(v)}{r_{\gamma+1}(u)} \left|y_\gamma(v) - s\right|^{k(v)} dv \right]^{\frac{1}{M}}.
\end{aligned}$$

Thus we have proved the inequality

$$[\delta^k_{\gamma+1}(u)]^{\frac{1}{M}} \le \left[\int_0^u \frac{r_\gamma(v)}{r_{\gamma+1}(u)} \left|y_{\gamma+1}(v) - s\right|^{k(v)} dv \right]^{\frac{1}{M}} + \left[\sigma^k_{\gamma+1}(u)\right]^{\frac{1}{M}}. \tag{2.11}$$

If $x(u) \to s[A_{\gamma+1}]_k$, then $x(u) \to s(A_{\gamma+1})$ due to Theorem 1 (ii), and thus the right side of inequality (2.11) tends to zero. Then also the left side of (2.11) tends to zero and therefore $\delta^k_{\gamma+1}(u) \to 0$ as $u \to \infty$.
(i) *Sufficiency*. Using the same technique as in the proof of necessity, we get the inequality

$$\left[\sigma^k_{\gamma+1}(u)\right]^{\frac{1}{M}} \le \left[\frac{1}{r_{\gamma+1}(u)} \int_0^u r_\gamma(v) \left|y_{\gamma+1}(v) - s\right|^{k(v)} dv \right]^{\frac{1}{M}} + [\delta^k_{\gamma+1}(u)]^{\frac{1}{M}}. \tag{2.12}$$

If $x(u) \to s(A_{\gamma+1})$ and $\left|y_{\gamma+1}(u) - y_\gamma(u)\right|^{k(u)} \to 0(C_{\gamma,\gamma+1})$, then it follows from (2.12) that $x \to 0[A_{\gamma+1}]_k$.

Statement (ii) can be proved in an analogous way with the help of (2.11) and (2.12) if $s = 0$. □

Remark 2. In particular, if $k(u) \equiv r$, Theorems 1–3 are formulated with some hints at proofs in [13] as Theorems 4–6. Theorems analogous to Theorems 1–3 for the matrix case are proved in [11] (as Theorems 4 and 5), where also references for partial cases can be found.

[2] We keep $\delta^k_{\gamma+1}(u)$ defined by (2.9) till the end of the paper.

3. COMPARATIVE ESTIMATIONS FOR SPEEDS OF $[A_{\alpha+1}]_k$-CONVERGENCE

Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ be a Riesz-type family and $k = k(u)$ be a positive function from X. Suppose that $\lambda = \lambda(u)$ is a positive function from X such that $\lambda(u) \to \infty$ as $u \to \infty$.

Definition 4. *We say that a function $x = x(u)$ is $[A_{\alpha+1}]_k$-convergent to s with speed λ if there exists the finite limit*

$$\lim_{u\to\infty} \lambda(u)\,\sigma^k_{\alpha+1}(u),$$

where $\sigma^k_{\alpha+1}(u)$ is defined by (2.1).

In this paper mainly the limit

$$\lim_{u\to\infty} \lambda(u)\,\sigma^k_{\alpha+1}(u) = 0$$

is used, but also the relation

$$\lambda(u)\,\sigma^k_{\alpha+1}(u) = O(1)$$

is used for describing the speed of $[A_{\alpha+1}]_k$-convergence of x.

The following Theorems 4–6 help us to estimate the speed of $[A_{\alpha+1}]_k$-convergence.

Theorem 4. *Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ be a Riesz-type family. Let there be given some positive function $\lambda = \lambda(u) \to \infty$ from X. Then the following statements are true for any $\gamma > \alpha_0$:*
(i) *if $[\lambda(u)]^M\,\sigma^k_{\gamma+1}(u) = o(1)$, then $\lambda(u)\,\sigma^{k'}_{\gamma+1}(u) = o(1)$, provided that $k(u)$ and $k'(u)$ satisfy the conditions $0 < k'(u) \le k(u) \le Mk'(u)$;*
(ii) *if $[\lambda(u)]^M\,\sigma^k_{\gamma+1}(u) = o(1)$, then $\lambda(u)[y_{\gamma+1}(u) - s] = o(1)$, provided that $1 \le k(u) \le \sup_u k(u) = M < \infty$.*

Proof. Take w.l.o.g. $s = 0$.
(i) By (2.3) we have the relation

$$\lambda(u)\,\sigma^{k'}_{\gamma+1}(u) = O(1)\{\lambda(u)\sigma^k_{\gamma+1}(u)\} + O(1)\{[\lambda(u)]^M\,\sigma^k_{\gamma+1}(u)\}^{\frac{1}{M}},$$

which implies statement (i) immediately.

(ii) By (2.4) and (2.6) we have the inequality

$$\lambda(u)\,\left|y_{\gamma+1}(u)\right| \le M_{\gamma,\gamma+1}\lambda(u)\,\sigma^1_{\gamma+1}(u).$$

Statement (i) and the last inequality complete the proof of (ii):

$$[\lambda(u)]^M\,\sigma^k_{\gamma+1}(u) \to 0 \implies \lambda(u)\,\sigma^1_{\gamma+1}(u) \to 0 \implies \lambda(u)\,y_{\gamma+1}(u) \to 0. \qquad \square$$

Remark 3. Theorem 4 remains true if we replace $o(1)$ by $O(1)$ everywhere in it.

In papers [14] and [15] the speeds λ_γ and λ_β of convergence $x = x(u)$ with respect to methods A_γ and A_β $(\beta > \gamma)$ are compared in a Riesz-type family (see [14], Theorem 1 and [15], Theorem 2). Speed $\lambda_\gamma = \lambda$ is supposed to be a given speed and $\lambda_\beta = \lambda_\beta(u)$ is defined by the relations

$$\lambda_{\beta+1}(u) = \frac{r_{\beta+1}(u)}{b_{\beta+1}(u)},\, b_{\beta+1}(u) = M_{\gamma,\beta}\int_0^u (u-v)^{\beta-\gamma-1} b_{\gamma+1}(v)\,dv,\, b_{\gamma+1}(u) = \frac{r_{\gamma+1}(u)}{\lambda(u)}. \tag{3.1}$$

Further we see that these speeds can be compared also for strong summability methods.

Theorem 5. *Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ be a Riesz-type family. Let there be given some function $\lambda = \lambda(u) \in X$ satisfying the condition $0 < \inf_u \lambda(u) \le \lambda(u) \to \infty$. Suppose that $k(u) \equiv r \ge 1$. Then we have for any $\beta > \gamma \underset{(-)}{>} \alpha_0$ the implication*

$$\lambda(u)\,\sigma^k_{\gamma+1}(u) = o(1) \Longrightarrow \lambda_{\beta+1}(u)\,\sigma^k_{\beta+1}(u) = o(1) \tag{3.2}$$

(as $u \to \infty$), provided that

$$\lim_{u\to\infty} \int_0^u b_{\gamma+1}(v)\,dv = \infty. \tag{3.3}$$

Proof. It follows from (2.8) that

$$\lambda_{\beta+1}(u)\,\sigma^r_{\beta+1}(u) = O(1)\left[\frac{\lambda_{\beta+1}(u)}{r_{\beta+1}(u)} \int_0^u (u-t)^{\beta-\gamma-1} \frac{r_{\gamma+1}(t)}{\lambda(t)} \lambda(t)\,\sigma^r_{\gamma+1}(t)\,dt\right].$$

Using formulas (3.1), we get

$$\lambda_{\beta+1}(u)\,\sigma^r_{\beta+1}(u) = O(1)\left[\frac{1}{b_{\beta+1}(u)} \int_0^u (u-t)^{\beta-\gamma-1}\, b_{\gamma+1}(t)\,\lambda(t)\,\sigma^r_{\gamma+1}(t)\,dt\right]. \tag{3.4}$$

The integral method $F_{\gamma,\beta}$, defined with the help of the function

$$f_{\gamma,\beta}(u,v) = \begin{cases} \frac{M_{\gamma,\beta}(u-v)^{\beta-\gamma-1} b_{\gamma+1}(v)}{b_{\beta+1}(u)} & \text{if } 0 \le v < u, \\ 0 & \text{if } v \ge u, \end{cases} \tag{3.5}$$

is regular by Proposition 1. That is why implication (3.2) follows from (3.4). □

Remark 4. Theorem 5 remains true if we replace $o(1)$ by $O(1)$ in (3.2). Comparative estimates for speeds λ and $\lambda_{\beta+1}$ defined through (3.1) can be found in [14] (Propositions 2, 3 and Examples 5–9). Note that $\lambda_{\beta+1}(u) \to \infty$ if $u \to \infty$ due to (3.3) (see Remark 2 in [15]).

Theorem 6. *Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ be a Riesz-type family and $0 < \lambda(u) \uparrow \infty$. Let for each $\gamma \underset{(-)}{>} \alpha_0$ the function $\lambda_{\gamma+1} = \lambda_{\gamma+1}(u)$ be defined with the help of the relations*

$$\lambda_{\gamma+1}(u) = \frac{r_{\gamma+1}(u)}{b_{\gamma+1}(u)} \text{ with } b_{\gamma+1}(u) = M_{\gamma,\gamma+1} \int_0^u b_\gamma(v)\,dv \text{ and } b_\gamma(u) = \frac{r_\gamma(u)}{[\lambda(u)]^{k(u)}}, \tag{3.6}$$

where (3.3) is satisfied with γ instead of $\gamma+1$. Then the following statements are true for any $\gamma \underset{(-)}{>} \alpha_0$ as $u \to \infty$:

(i) *if $\lambda(u)[y_\gamma(u) - s] = o(1)$, then $\lambda_{\gamma+1}(u)\,\sigma^k_{\gamma+1}(u) = o(1)$, provided that $\inf_u k(u) = m > 0$;*

(ii) *if $[\lambda(u)]^M \sigma^k_{\gamma+1}(u) = o(1)$, then $\lambda_{\gamma+1}(u)\,\delta^k_{\gamma+1}(u) = o(1)$, provided that $1 \le k(u) \le \sup_u k(u) = M < \infty$;*

(iii) *if $\lambda_{\gamma+1}(u)\,\delta^k_{\gamma+1}(u) = o(1)$ and $\lambda(u)\left[y_{\gamma+1}(u) - s\right] = o(1)$, then $\lambda_{\gamma+1}(u)\,\sigma^k_{\gamma+1}(u) = o(1)$, provided that $1 \le k(u) \le \sup_u k(u) = M < \infty$.*

Proof.

(i) By (2.1) and (3.6) we have

$$\begin{aligned} \lambda_{\gamma+1}(u)\,\sigma^k_{\gamma+1}(u) &= \frac{\lambda_{\gamma+1}(u)}{r_{\gamma+1}(u)} \int_0^u \frac{r_\gamma(v)}{[\lambda(v)]^{k(v)}} \left|\lambda(v)\left[y_\gamma(v) - s\right]\right|^{k(v)} dv \\ &= \frac{1}{b_{\gamma+1}(u)} \int_0^u b_\gamma(v) \left|\lambda(v)\left[y_\gamma(v) - s\right]\right|^{k(v)} dv. \end{aligned}$$

Statement (i) follows from the last equality, because the integral method $F_{\gamma,\gamma+1}$ defined by (3.5) is regular.

(ii) We realize that $\lambda(u) \geq 1$ and therefore $\lambda_{\gamma+1}(u) \leq [\lambda(u)]^M$ for sufficiently large u. As $\delta^k_{\gamma+1}(u)$ is defined by (2.9), we get from (2.11) with the help of (3.6) for sufficiently large u:

$$\begin{aligned}\left[\lambda_{\gamma+1}(u)\,\delta^k_{\gamma+1}(u)\right]^{\frac{1}{M}} &\leq \left[\frac{\lambda_{\gamma+1}(u)}{r_{\gamma+1}(u)}\int_0^u \frac{r_\gamma(v)}{[\lambda(v)]^{k(v)}}\left|\lambda(v)\left[y_{\gamma+1}(v)-s\right]\right|^{k(v)} dv\right]^{\frac{1}{M}} \\ &\quad + \left[\lambda_{\gamma+1}(u)\,\sigma^k_{\gamma+1}(u)\right]^{\frac{1}{M}} \\ &\leq \left\{\frac{1}{b_{\gamma+1}(u)}\int_0^u b_\gamma(v)\left|\lambda(v)\left[y_{\gamma+1}(v)-s\right]\right|^{k(v)} dv\right\}^{\frac{1}{M}} \\ &\quad + \left\{[\lambda(u)]^M\,\sigma^k_{\gamma+1}(u)\right\}^{\frac{1}{M}}.\end{aligned}$$

Statement (ii) follows now immediately from the last inequality, because the method $F_{\gamma,\gamma+1}$ defined by (3.5) is regular and $[\lambda(u)]^M\,\sigma^k_{\gamma+1}(u) \to 0$ implies $\left|\lambda(u)\left[y_{\gamma+1}(u)-s\right]\right|^{k(u)} \to 0$ by Theorem 4 (ii).

(iii) Starting from (2.12), we prove the inequalities which imply (iii) immediately:

$$\begin{aligned}\left[\lambda_{\gamma+1}(u)\,\sigma^k_{\gamma+1}(u)\right]^{\frac{1}{M}} &\leq \left[\frac{\lambda_{\gamma+1}(u)}{r_{\gamma+1}(u)}\int_0^u \frac{r_\gamma(v)}{[\lambda(v)]^{k(v)}}\left|\lambda(v)\left[y_{\gamma+1}(v)-s\right]\right|^{k(v)} dv\right]^{\frac{1}{M}} \\ &\quad + \left[\lambda_{\gamma+1}(u)\,\delta^k_{\gamma+1}(u)\right]^{\frac{1}{M}} \\ &\leq \left[\frac{1}{b_{\gamma+1}(u)}\int_0^u b_\gamma(v)\left|\lambda(v)\left[y_{\gamma+1}(v)-s\right]\right|^{k(v)} dv\right]^{\frac{1}{M}} \\ &\quad + \left[\lambda_{\gamma+1}(u)\,\delta^k_{\gamma+1}(u)\right]^{\frac{1}{M}}.\end{aligned}$$

□

Remark 5. Theorem 6 remains true if we replace $o(1)$ by $O(1)$ everywhere in it.

4. COMPARISON OF $[A_{\alpha+1}]_k$-CONVERGENCE WITH STATISTICAL CONVERGENCE

Here we compare $[A_{\alpha+1}]_k$-convergence $(\alpha \underset{(-)}{>} \alpha_0)$ of $x = x(u)$ with its A-statistical convergence, where the method $A = C_{\alpha,\alpha+1}$ is defined by (1.6).

Theorem 7. *Let* $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ *be a Riesz-type family satisfying* (1.8) *if* α_0 *is included. Suppose that* $\sup_u k(u) = M < \infty$. *Then the following statements are true for functions* $x = x(u)$ *and numbers* s *and* $\gamma \underset{(-)}{>} \alpha_0$:

(i) *if* $x(u) \to s[A_{\gamma+1}]_k$, *then* $A_\gamma x \to s(st_{C_{\gamma,\gamma+1}})$;

(ii) *if* $x(u) = O(A_\gamma)$ *and* $A_\gamma x \to s(st_{C_{\gamma,\gamma+1}})$, *then* $x(u) \to s[A_{\gamma+1}]_k$, *provided that* $\inf_u k(u) = m > 0$.

Proof. Choose an arbitrary $\varepsilon > 0$. According to (1.2) we denote

$$\mathscr{K}_{\varepsilon,u} = \{v \in [0,u] : |y_\gamma(v) - s| \geq \varepsilon\}.$$

(i) We get for $\sigma^k_{\gamma+1}(u)$ defined by (2.1) the inequalities

$$\sigma^k_{\gamma+1}(u) \geq \int_{\mathscr{K}_{\varepsilon,u}} \frac{r_\gamma(v)}{r_{\gamma+1}(u)}\left|y_\gamma(v)-s\right|^{k(v)} dv \geq h(\varepsilon)\int_{\mathscr{K}_{\varepsilon,u}} \frac{r_\gamma(v)}{r_{\gamma+1}(u)}\, dv,$$

where $h(\varepsilon)=\min\{1,\varepsilon^M\}$. Therefore, if $\sigma^k_{\gamma+1}(u)\to 0$ (as $u\to\infty$), then also the integral in the right side of the last inequalities tends to zero, i.e., $y_\gamma(u)\to s(st_{C_{\gamma,\gamma+1}})$. That proves (i).

(ii) Denoting $\mathscr{K}^*_{\varepsilon,u}=\{v\in[0,u]:|y_\gamma(v)-s|<\varepsilon\}$, we get

$$\begin{aligned}\sigma^k_{\gamma+1}(u) &= \int_{\mathscr{K}_{\varepsilon,u}}\frac{r_\gamma(v)}{r_{\gamma+1}(u)}\left|y_\gamma(v)-s\right|^{k(v)}dv+\int_{\mathscr{K}^*_{\varepsilon,u}}\frac{r_\gamma(v)}{r_{\gamma+1}(u)}\left|y_\gamma(v)-s\right|^{k(v)}dv\\ &\le (L+|s|)^M\int_{\mathscr{K}_{\varepsilon,u}}\frac{r_\gamma(v)}{r_{\gamma+1}(u)}dv+H(\varepsilon)\int_0^u\frac{r_\gamma(v)}{r_{\gamma+1}(u)}dv,\end{aligned}$$

where $|y_\gamma(u)|\le L$ and $H(\varepsilon)=\max\{\varepsilon^m,\varepsilon^M\}$. If $u\to\infty$, then

$$\lim_{u\to\infty}\sigma^k_{\gamma+1}(u)\le(L+|s|)^M\lim_{u\to\infty}\int_{\mathscr{K}_{\varepsilon,u}}\frac{r_\gamma(v)}{r_{\gamma+1}(u)}dv+H(\varepsilon)/M_{\gamma,\gamma+1},$$

where $M_{\gamma,\gamma+1}$ is defined in Definition 2. If $A_\gamma x\to s(st_{C_{\gamma,\gamma+1}})$, i.e., if the limit in the right side of the last inequality is zero, then

$$\lim_{u\to\infty}\sigma^k_{\gamma+1}(u)\le H(\varepsilon)/M_{\gamma,\gamma+1}.$$

As $\varepsilon>0$ is arbitrarily chosen, the last inequality implies that $\sigma^k_{\gamma+1}(u)=o(1)$ as $u\to\infty$. □

In particular, if we consider the family of Riesz methods $A_\alpha=(R,\alpha)$ $(\alpha\ge 0)$ and $k(u)\equiv r$, then Theorem 7 gives for $\gamma=0$ Theorem 2 from [3].

In order to see how statistical convergence is related to ordinary convergence in statements (i) and (ii) of Theorem 7, we formulate the proposition which can be proved in the same way as Theorem 7 (take $k(u)\equiv 1$ in its proof).

Proposition 2. *Let* $\{A_\alpha\}$ $(\alpha\underset{(-)}{>}\alpha_0)$ *be a Riesz-type family satisfying* (1.8) *if* α_0 *is included. Then the following statements are true for functions* $x=x(u)$ *and numbers* s *and* $\gamma\underset{(-)}{>}\alpha_0$:

(i) *if* $x(u)\to s(A_\gamma)$, *then* $A_\gamma x\to s(st_{C_{\gamma,\gamma+1}})$;

(ii) *if* $x(u)=O(A_\gamma)$ *and* $A_\gamma x\to s(st_{C_{\gamma,\gamma+1}})$, *then* $x(u)\to s(A_{\gamma+1})$.

5. CONCLUSIONS

In this paper a Riesz-type family of summability methods A_α $(\alpha\underset{(-)}{>}\alpha_0)$ is considered (see Definition 2). The strong summability methods $[A_{\alpha+1}]_k$ are defined (see Definition 3) and described with the help of inclusion theorems. These theorems give the conditions for comparing the methods $[A_{\alpha+1}]_k$ with each other and with methods A_α (for different values of α) by summability fields (see Theorems 1–3) and by speed of convergence (see Theorems 4–6). The methods $[A_{\alpha+1}]_k$ are compared also with certain methods of statistical convergence (see Theorem 7). Theorems 1–3 generalize the theorems known earlier, in particular, for the case $k=k(u)\equiv r$ (see [13]), showing that the methods $[A_{\alpha+1}]_k$ defined here are more flexible. In the authors' view the notion of methods $[A_{\alpha+1}]_k$ can be further generalized with the help of a modulus function f. A convexity theorem can also be proved for these methods.

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Tugeva summeeruvuse menetlused Rieszi tüüpi peres

Anna Šeletski ja Anne Tali

Autorid jätkavad summeerimismenetluste Rieszi tüüpi perede $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_0)$ uurimist (vt [14] ja [15]). On vaadeldud menetluste $A_{\alpha+1}$ jaoks defineeritud tugeva summeeruvuse menetlusi $[A_{\alpha+1}]_k$. On tõestatud sisalduvusteoreemid, mis lubavad vaadeldavaid tugeva summeeruvuse menetlusi võrrelda (parameetri α erinevate väärtuste korral) omavahel ja menetlustega perest $\{A_\alpha\}$ nii summeerimisväljade kui -kiiruste järgi. Tugeva summeeruvuse menetlusi on võrreldud ka teatavate statistilise koonduvuse menetlustega.