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MATHEMATICS

# Approximation in variation by the Kantorovich operators

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**Abstract.** We discuss the rate of approximation of the Kantorovich operators. The rate of approximation is given with respect to the variation seminorm.

**Key words:** approximation theory, Kantorovich operators, convergence in variation, order of approximation, absolutely continuous functions.

## 1. INTRODUCTION

The paper deals with the convergence in variation of the Kantorovich operators. In [7] we proved similar results for the Meyer-König and Zeller operators.

Let $TV[0,1]$, respectively $AC[0,1]$, denote the class of all functions of bounded variation, respectively the absolutely continuous functions on $[0,1]$. Let $L_n : TV[0,1] \to TV[0,1]$ be an arbitrary positive operator, i.e. for $f \geq 0$ we have $L_n f \geq 0$. It is known that many linear positive operators have the variation detracting property (or the variational diminishing property, cf. [10]) in the following form: for all $f \in TV[0,1]$ with the total variation $V_{[0,1]}[f]$ we have $L_n f \in TV[0,1]$ and

$$V_{[0,1]}[L_n f] \leq V_{[0,1]}[f].$$

For example, the variation detracting property is valid for the Bernstein, Meyer-König and Zeller, and Stancu operators (see [2], where the problem is posed and solved even for the $\varphi$-variation). The variation detracting property is needed to consider the convergence in variation, i.e. for all $f \in TV[0,1]$ there has to be

$$V_{[0,1]}[L_n f - f] \to 0.$$

The convergence in $\varphi$-variation of many positive operators was considered in [2], but not of Kantorovich-type operators.

The operators of Kantorovich

$$(K_n f)(x) = (n+1) \sum_{k=0}^{n} p_{k,n}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du \quad (x \in [0,1]), \tag{1.1}$$

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where

$$p_{k,n}(x) := \binom{n}{k} x^k (1-x)^{n-k},$$

were introduced in [6] and their asymptotic or approximation behaviour has been investigated in many works (see, for example, [1,4,5,8] and literature cited there). In the case of the Kantorovich operators the variation detracting property holds as follows.

**Theorem A.** ([3], Proposition 3.3) *If* $f \in TV[0,1]$, *then* $K_n f \in AC[0,1]$ *and*

$$V_{[0,1]}[K_n f] \le V_{[0,1]}[f].$$

The convergence in variation for smooth functions is given in Section 2, Theorem 1. We prove not only the convergence in variation of the Kantorovich operators, but also give the rate of approximation in Theorem 2.

For the following proof of Theorem 1 we calculate the derivative of $K_n f$,

$$(K_n f)'(x) = \frac{n+1}{X} \sum_{k=0}^{n} (k - nx) p_{k,n}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du \quad (x \in (0,1)), \tag{1.2}$$

where $X = x(1-x)$. In the same proof we need the sum moments for the operators (1.1). Let us define the sum moments as in [3]:

$$T_{r,n}(x) := \sum_{k=0}^{n} [k - nx]^r p_{k,n}(x). \tag{1.3}$$

Then there hold the identities (see, e.g., [3] or original presentation in [9])

$$T_{r,n}(x) = \begin{cases} 1, & r = 0, \\ 0, & r = 1, \\ nX, & r = 2, \\ nX(1-2x), & r = 3. \end{cases} \tag{1.4}$$

## 2. APPROXIMATION IN VARIATION BY THE KANTOROVICH OPERATORS

We start with studying the rate of approximation of smooth functions with respect to the variation seminorm.

**Theorem 1.** *If* $g'' \in AC[0,1]$, *then*

$$V_{[0,1]}[K_n g - g] \le \frac{4}{n+1} (V_{[0,1]}[g] + V_{[0,1]}[g'']) \qquad (n \ge 3). \tag{2.1}$$

*Proof.* We represent $f$ in (1.2) by Taylor's formula with the integral remainder term

$$g(t) = g(x) + (t-x)g'(x) + (t-x)^2 \frac{g''(x)}{2} + \frac{1}{2} \int_x^t (t-v)^2 g'''(v)dv.$$

We have

$$(K_n g)'(x) = A_{0,n}(x) g(x) + A_{1,n}(x) g'(x) + A_{2,n}(x) \frac{g''(x)}{2} + (R_n g)(x), \tag{2.2}$$

where

$$A_{j,n}(x)=\frac{n+1}{X}\sum_{k=0}^{n}(k-nx)p_{k,n}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^j dt \quad (j=0,1,2) \tag{2.3}$$

and

$$(R_n g)(x):=\frac{n+1}{2X}\sum_{k=0}^{n}(k-nx)p_{k,n}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}dt\int_x^t (t-v)^2 g'''(v)dv. \tag{2.4}$$

Calculating (2.3) by (1.3) and (1.4), we have

$$A_{0,n}(x)=\frac{T_{1,n}(x)}{X}=0.$$

Analogously,

$$A_{1,n}(x)=\frac{T_{2,n}(x)+(\frac{1}{2}-x)T_{1,n}(x)}{(n+1)X}=\frac{n}{n+1}$$

and

$$A_{2,n}(x)=\frac{T_{3,n}(x)+(1-2x)T_{2,n}(x)+(\frac{1}{3}-x+x^2)T_{1,n}(x)}{(n+1)^2X}=\frac{2n(1-2x)}{(n+1)^2}.$$

So, by (2.2), (2.3) and (2.4) we have for the derivative

$$(K_n g)'(x)=\frac{n}{n+1}g'(x)+\frac{n(1-2x)}{(n+1)^2}g''(x)+(R_n g)(x). \tag{2.5}$$

The integration domain of the double integral in the remainder (2.4) is

$$D_{x,k}=\Big\{(t,v)\Big|\ \frac{k}{n+1}\le t\le\frac{k+1}{n+1},\ v\in[x,t]\Big\}.$$

We denote $t_k:=\frac{k}{n+1}$. For fixed $x\in(0,1)$ we divide the summation indices $k$ into three sets: $t_{k+1}\le x$, $t_k\le x<t_{k+1}$ or $t_k>x$. Hence, for the remainder we get ($[x]$ denotes the integer part of $x$)

$$\begin{aligned}(R_n g)(x) &= \sum_{t_{k+1}\le x}\ldots+\sum_{t_k\le x<t_{k+1}}\ldots+\sum_{t_k>x}\ldots\\ &= \sum_{k=0}^{[(n+1)x]-1}\ldots+\sum_{k=[(n+1)x]}^{[(n+1)x]}\ldots+\sum_{k=[(n+1)x]+1}^{n}\ldots.\end{aligned}$$

Here and later on we take $\sum_{k=0}^{a}\ldots=0$ if $a<0$ and $\sum_{k=b}^{n}\ldots=0$ if $b>n$. In each summand we change the order of integration by splitting the double integration domain $D_{x,k}$ in a suitable way. After that we get six different sums. So we have

$$(R_n g)(x)=\sum_{i=1}^{6}B_{i,n}g(x)\equiv\sum_{i=1}^{6}B_{i,n}g, \tag{2.6}$$

where, denoting

$$q_{k,n}(x):=\frac{n+1}{2X}(k-nx)p_{k,n}(x),$$

we obtain

$$B_{1,n}g := -\sum_{k=0}^{[(n+1)x]-1} q_{k,n}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} g'''(v)dv\int_{\frac{k}{n+1}}^{v}(t-v)^2dt, \tag{2.7}$$

$$B_{2,n}g := -\sum_{k=0}^{[(n+1)x]-1} q_{k,n}(x)\int_{\frac{k+1}{n+1}}^{x} g'''(v)dv\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-v)^2dt, \tag{2.8}$$

$$B_{3,n}g := -\sum_{k=[(n+1)x]}^{[(n+1)x]} q_{k,n}(x)\int_{\frac{k}{n+1}}^{x} g'''(v)dv\int_{\frac{k}{n+1}}^{v}(t-v)^2dt, \tag{2.9}$$

$$B_{4,n}g := \sum_{k=[(n+1)x]}^{[(n+1)x]} q_{k,n}(x)\int_{x}^{\frac{k+1}{n+1}} g'''(v)dv\int_{v}^{\frac{k+1}{n+1}}(t-v)^2dt, \tag{2.10}$$

$$B_{5,n}g := \sum_{k=[(n+1)x]+1}^{n} q_{k,n}(x)\int_{x}^{\frac{k}{n+1}} g'''(v)dv\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-v)^2dt, \tag{2.11}$$

$$B_{6,n}g := \sum_{k=[(n+1)x]+1}^{n} q_{k,n}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} g'''(v)dv\int_{v}^{\frac{k+1}{n+1}}(t-v)^2dt. \tag{2.12}$$

Let us estimate $B_{1,n}g$ in (2.7). Since for $\frac{k}{n+1} \le v \le \frac{k+1}{n+1}$ we have

$$\int_{\frac{k}{n+1}}^{v}(t-v)^2dt \le \frac{1}{3(n+1)^3},$$

by using Cauchy's inequality and (1.3), we get for (2.7) the estimate

$$\begin{aligned}|(B_{1,n}g)(x)| &\le \frac{\|g'''\|}{6X(n+1)^2}\left(\sum_{k=0}^{n}(k-nx)^2p_{k,n}(x)\right)^{\frac12}\left(\sum_{k=0}^{n}p_{k,n}(x)\right)^{\frac12}\\ &\le \frac{1}{6X(n+1)^2}\,(T_{2,n}(x))^{\frac12}\,(T_{0,n}(x))^{\frac12}\,\|g'''\|,\end{aligned}$$

where here and later on the norm is taken in $L^1(0,1)$, i.e.

$$\|f\| := \|f\|_{L^1(0,1)}.$$

Finally, by (1.4) and $\int_0^1 \frac{1}{\sqrt{X}}dx = \pi$ we obtain for norms

$$||B_{1,n}g|| \le \frac{\pi}{6(n+1)^{\frac32}}||g'''||. \tag{2.13}$$

We get in a similar way the same estimate for $B_{6,n}g$ in (2.12).

In the following we estimate the sum $B_{2,n}g + B_{5,n}g$ of expressions in (2.8) and (2.11). Let by (2.8)

$$B_{2,n}g = B^1_{2,n}g + B^2_{2,n}g := \sum_{k=0}^{[(n+1)x]-1} \dots \int_{x}^{\frac{k}{n}} \dots dv \dots + \sum_{k=0}^{[(n+1)x]-1} \dots \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} \dots dv \dots \tag{2.14}$$

and by (2.11)

$$B_{5,n}g = B^1_{5,n}g - B^2_{5,n}g := \sum_{k=[(n+1)x]+1}^{n} \dots \int_{x}^{\frac{k}{n}} \dots dv \dots - \sum_{k=[(n+1)x]+1}^{n} \dots \int_{\frac{k}{n+1}}^{\frac{k}{n}} \dots dv \dots . \tag{2.15}$$

In expressions of $B_{2,n}^2 g$ and $B_{5,n}^2 g$ the variable $v$ is on $[\frac{k}{n+1}, \frac{k+1}{n+1}]$, therefore

$$\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt \le \frac{1}{3(n+1)^3}, \quad v \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]. \tag{2.16}$$

Hence, the quantities $B_{2,n}^2 g$ and $B_{5,n}^2 g$ can be estimated in the same way as we did before for $B_{1,n} g$. So we can state that the estimate (2.13) is valid also for $B_{2,n}^2 g$ and $B_{5,n}^2 g$. Now consider the sum

$$\begin{aligned}
B_{2,n}^1 g + B_{5,n}^1 g \quad &:= \sum_{k=0}^{[(n+1)x]-1} q_{k,n}(x) \int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt \\
&\quad + \sum_{k=[(n+1)x]+1}^{n} q_{k,n}(x) \int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt \\
&= \sum_{k=0}^{n} q_{k,n}(x) \int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt \\
&\quad - \sum_{k=[(n+1)x]}^{[(n+1)x]} q_{k,n}(x) \int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt \\
&=: C_{1,n} g - C_{2,n} g.
\end{aligned}$$

For $C_{2,n} g$ we can prove that (2.13) is valid. Indeed, for fixed $x \in (0,1)$ and $n \in I\!N$ the quantity $C_{2,n} g$ consists of one summand with the index $k = [(n+1)x]$, i.e.

$$C_{2,n} g(x) = q_{k,n}(x) \int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt. \tag{2.17}$$

To estimate the norm $\|C_{2,n} g\|$, we decompose the interval $(0,1)$ into $n+1$ equal parts,

$$\begin{aligned}
\|C_{2,n} g\| \quad &= \frac{n+1}{2} \sum_{i=0}^{n} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \frac{|k-nx|}{x(1-x)} p_{k,n}(x) \Big| \int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-v)^2 dt \Big| dx \\
&\le \frac{n+1}{2} \sum_{i=0}^{n} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \frac{|i-nx|}{x(1-x)} p_{i,n}(x) \int_x^{\frac{i}{n}} |g'''(v)|dv \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} (t-v)^2 dt dx.
\end{aligned}$$

Here the second relation holds because for $\frac{i}{n+1} \le x < \frac{i+1}{n+1}$ and $k = [(n+1)x]$ we have $k = i$. By (2.16) and Cauchy's inequality we get

$$\begin{aligned}
\|C_{2,n} g\| \quad &\le \frac{\|g'''\|}{6(n+1)^2} \sum_{i=0}^{n} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \frac{|i-nx|}{X} p_{i,n}(x) dx \\
&\le \frac{\|g'''\|}{6(n+1)^2} \int_0^1 \frac{1}{X} \Big( \sum_{i=0}^{n} |i-nx|^2 p_{i,n}(x) \Big)^{\frac{1}{2}} \Big( \sum_{i=0}^{n} p_{i,n}(x) \Big)^{\frac{1}{2}} dx,
\end{aligned}$$

which by (1.4) yields

$$\|C_{2,n} g\| \le \frac{\|g'''\|}{6(n+1)^2} \int_0^1 \frac{(nX)^{\frac{1}{2}}}{X} dx \le \frac{\pi \|g'''\|}{6(n+1)^{\frac{3}{2}}}.$$

Thus, for $C_{2,n} g$ in (2.17) again (2.13) is valid.

Looking at (2.17) and (2.9), (2.10), it is clear that also $B_{3,n}g$ and $B_{4,n}g$ can be estimated as in (2.13).

The summary of the estimates for $B_{i,n}g \quad (i=1,\dots,6)$ in (2.7)–(2.12) states:

1. for $B_{i,n}g$ with $i=1,3,4,6$ the estimate (2.13) is valid,
2. for the sum $B_{2,n}g+B_{5,n}g \equiv (B^1_{2,n}g+B^2_{2,n}g)+(B^1_{5,n}g+B^2_{5,n}g)$ the terms $B^2_{2,n}g$ and $B^2_{5,n}g$ satisfy (2.13), and in $B^1_{2,n}g+B^1_{5,n}g \equiv C_{1,n}g+C_{2,n}g$ the term $C_{2,n}g$ satisfies (2.13) as well. So, we can write

$$\begin{aligned}
||B_{i,n}g|| &\leq \frac{\pi}{6(n+1)^{\frac{3}{2}}}||g'''||, \quad i=1,3,4,6,\\
||B^2_{j,n}g|| &\leq \frac{\pi}{6(n+1)^{\frac{3}{2}}}||g'''||, \quad j=2,5, \qquad (2.18)\\
||C_{2,n}g|| &\leq \frac{\pi}{6(n+1)^{\frac{3}{2}}}||g'''||.
\end{aligned}$$

Finally, we have to estimate

$$(C_{1,n}g)(x) := \frac{n+1}{2X}\sum_{k=0}^{n}(k-nx)p_{k,n}(x)\int_x^{\frac{k}{n}} g'''(v)dv \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-v)^2 dt.$$

As $k-nx$ and the integral $\int_x^{\frac{k}{n}}|\dots|$ have the same sign, we get

$$\begin{aligned}
|(C_{1,n}g)(x)| \leq\ & \frac{1}{6X}\sum_{k=0}^{n}(k-nx)p_{k,n}(x)\int_x^{\frac{k}{n}}|g'''(v)|\Bigg(\left(\frac{k+1}{n+1}-v\right)^2\\
&+\left(\frac{k+1}{n+1}-v\right)\left(\frac{k}{n+1}-v\right)+\left(\frac{k}{n+1}-v\right)^2\Bigg)dv.
\end{aligned}$$

The function

$$h(v) := \left(\frac{k+1}{n+1}-v\right)^2+\left(\frac{k+1}{n+1}-v\right)\left(\frac{k}{n+1}-v\right)+\left(\frac{k}{n+1}-v\right)^2$$

is positive, convex everywhere and $h(\frac{k}{n}) \leq \frac{1}{(n+1)^2}$. Therefore, by an obvious inequality $h(v) \leq h(x)+h(\frac{k}{n})$ for $v\in[x,\frac{k}{n}]$ we obtain

$$\begin{aligned}
\|C_{1,n}g\| \leq\ & \int_0^1 \frac{1}{6X}\sum_{k=0}^{n}(k-nx)p_{k,n}(x)\left(\int_0^{\frac{k}{n}}-\int_0^x\right)|g'''(v)|dv\\
&\times\left(h(x)+\frac{1}{(n+1)^2}\right)dx =: C^1_{1,n}-C^2_{1,n}. \qquad (2.19)
\end{aligned}$$

For integration we represent $h(x)$ by powers of $k-nx$ in the form

$$(n+1)^2h(x) = 3(k-nx)^2+3(1-2x)(k-nx)+(1-3X).$$

Applying Fubini's theorem for the first part of (2.19), we get

$$C^1_{1,n} := \frac{1}{6(n+1)^2}\sum_{k=0}^{n}\int_0^{\frac{k}{n}}|g'''(v)|dv\int_0^1\frac{((n+1)^2h(x)+1)(k-nx)}{X}p_{k,n}(x)dx,$$

where the inner integral can be evaluated and estimated by

$$\begin{aligned} I_{k,n} &:= \Big|\int_0^1 \frac{3(k-nx)^3+3(1-2x)(k-nx)^2+(2-3X)(k-nx)}{X} p_{k,n}(x)dx\Big| \\ &= \Big|\frac{3(2k-n)}{n+2}\Big| \le \frac{3n}{n+2}, \quad 1 \le k \le n-1 \end{aligned}$$

and

$$I_{k,n} \le \frac{3n}{n+2} + 2, \quad k=0,\, k=n.$$

Hence, summing the former integral over all $k$ yields

$$\begin{aligned} |C^1_{1,n}| &\le \frac{1}{6(n+1)^2}\left(\frac{3n(n+1)}{n+2}+4\right)\|g'''\| \\ &\le \frac{1}{2(n+1)}\|g'''\|, \quad n \ge 1. \end{aligned} \tag{2.20}$$

We calculate the second part of (2.19) using the representation of $(n+\ 1)^2 h(x)$, and the moments (1.3) and (1.4),

$$\begin{aligned} C^2_{1,n} &:= \int_0^1 \frac{1}{6X}\int_0^x |g'''(v)|dv \sum_{k=0}^n (k-nx)p_{k,n}(x)\left(h(x)+\frac{1}{(n+1)^2}\right)dx \\ &= \int_0^1 \frac{1}{6X(n+1)^2}\int_0^x |g'''(v)|dv\,(6nX(1-2x))\,dx \\ &= \frac{n}{(n+1)^2}\int_0^1 (v^2-v)|g'''(v)|dv, \end{aligned}$$

hence

$$|C^2_{1,n}| \le \frac{1}{4(n+1)}\|g'''\|. \tag{2.21}$$

Therefore, for (2.19) we obtain

$$\|C_{1,n}g\| \le \frac{3}{4(n+1)}\|g'''\|. \tag{2.22}$$

Collecting the results in (2.18) and (2.22), we have by (2.6)

$$||R_n g|| \le \frac{9+7\pi}{12(n+1)}||g'''|| \qquad (n \ge 3).$$

Finally, we obtain by using (2.5)

$$||(K_n g)'-g'|| \le \frac{||g'||}{n+1}+\frac{||g''||}{n+1}+\frac{9+7\pi}{12(n+1)}||g'''|| \qquad (n \ge 3).$$

Now we use Stein's inequality with the exact constant (see, e.g., [11], Theorem A10.1)

$$||g''||_{L^1} \le \frac{\pi^3}{16}\sqrt{||g'||_{L^1}||g'''||_{L^1}}$$

and the inequality for the geometric and arithmetic means. So we have

$$||(K_n g)'-g'|| \le \frac{4}{n+1}(||g'||+||g'''||) \qquad (n \ge 3),$$

which finishes our proof. □

The proof of Theorem 3.7 in [3] (see also [7]) gives a general scheme for getting the rate of convergence of absolutely continuous functions from the corresponding convergence theorem of smooth functions (Theorem 1). Since Theorem A and Theorem 1 are valid for the Kantorovich operators, we may formulate the general result by using this scheme. Below $(\overline{\triangle}_h^r g)(x)$ denotes the central difference of $g$ of order $r$,

$$(\overline{\triangle}_h^r g)(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} g\left(x + \left(\frac{r}{2} - k\right) h\right).$$

**Theorem 2.** *If $f \in AC[0,1]$, then there exist constants $c_1, c_2 > 0$ such that*

$$V_{[0,1]}[K_n f - f] \leq c_1 \sup_{0<h\leq n^{-\frac{1}{2}}} V_{[h,1-h]}[\overline{\triangle}_h^2 f] + c_2 \frac{V_{[0,1]}[f]}{n}.$$

*In particular, if $f' \in AC[0,1]$, then there exists a constant $c_3 > 0$ such that*

$$V_{[0,1]}[K_n f - f] \leq \frac{c_3}{\sqrt{n}} \sup_{0<h\leq n^{-\frac{1}{2}}} V_{[\frac{h}{2},1-\frac{h}{2}]}[\overline{\triangle}_h^1 f'] + c_2 \frac{V_{[0,1]}[f]}{n}.$$

## 3. CONCLUSIONS

Lorentz [9] was the first to consider the variation detracting property for the Bernstein operators. Because the Bernstein operators are classical prototypes for many positive operators, the variation detracting property has been studied for many positive operators like the Meyer-König and Zeller, Kantorovich, Stancu operators, etc. In this paper the variation detracting property is related in a natural way with the convergence in variation, particularly for the Kantorovich operators (Theorems 1 and 2). The study of the convergence in variation seminorm is a comparatively new field in the theory of approximation (see, e.g., [3]). Our Theorems 1 and 2 solve the problem of the convergence in variation for the Kantorovich operators. The proof of the main theorem, Theorem 1, follows the known idea using Taylor's formula, but it is nontrivial in any meaning. Probably, some technical tricks from our proof can be used for some other positive operators.

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## Kantorovichi operaatoritega lähendamine variatsiooni mõttes

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On uuritud Kantorovichi operaatoritega lähendamise kiirust, mida mõõdetakse funktsiooni (või selle tuletiste) täisvariatsiooni abil. On tõestatud, et kui lähendatav funktsioon on absoluutselt pidev (või vastavalt esimene ehk teine tuletis on absoluutselt pidevad), siis lähendamise kiirus on hinnatav funktsiooni (vastavalt selle tuletiste) täisvariatsiooni kaudu.