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MATHEMATICS

# Group actions, orbit spaces, and noncommutative deformation theory

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**Abstract.** Consider the action of a group $G$ on an ordinary commutative $k$-variety $X = \mathrm{Spec}(A)$. In this note we define the category of $A$–$G$-modules and their deformation theory. We then prove that this deformation theory is equivalent to the deformation theory of modules over the noncommutative $k$-algebra $A[G] = A\sharp G$. The classification of orbits can then be studied over a commutative ring, and we give an example of this on surface cyclic singularities.

**Key words:** $A$–$G$ module, noncommutative deformation theory, noncommutative blowup, cyclic surface singularities, orbit closures, swarm of modules, $r$-pointed artinian $k$-algebras, noncommutative deformation functor, Generalized Matric Massey Products (GMMP), McKay correspondence.

## 1. INTRODUCTION

Consider the action of a group $G$ on an ordinary commutative $k$-variety $X = \mathrm{Spec}(A)$. We define the category of $A$–$G$-modules, Definition 2.1, and their deformation theory. We then prove that this deformation theory is equivalent to the deformation theory of modules over the noncommutative $k$-algebra $A[G] = A\sharp G$. Thus the noncommutative moduli of the one-sided $A[G]$-modules can be computed as the noncommutative moduli of $A$-modules with $A$ commutative, invariant under the (dual) action of the group $G$, which simplify the computations significantly. The orbit closure of $x \in X$ corresponds to an $A[G]$-module $M_x = A/\mathfrak{a}_x$, so that the classification of closures of orbits can be studied locally by deformation theory of $M_x$ as an $A$–$G$-module. Finally, we work through an example of the noncommutative blowup of cyclic surface singularities.

## 2. MODULES WITH GROUP ACTIONS

Let $k$ be an algebraically closed field of characteristic 0. Let $G$ be a finite dimensional reductive algebraic group acting on an affine scheme $X = \mathrm{Spec}\,A$, $A$ a finitely generated (commutative) $k$-algebra. Let $\mathfrak{a}_x$ be the ideal of the closure of the orbit of $x$ and let $G \to \mathrm{Aut}_k(A)$ sending $g$ to $\nabla_g$ be the induced action of $G$ on $A$. Then, as the ideal $\mathfrak{a}_x$ is invariant under the action of $G$ on $A$, we get an induced action on $A/\mathfrak{a}_x$. The skew group algebra over $A$ is denoted $A[G]$. It consists of all formal sums $\sum\limits_{g\in G} a_g g$ with product defined by

$$(a_1g_1)(a_2g_2) = a_1\nabla_{g_1}(a_2)g_1g_2.$$

For later use notice that this definition extends the definition of the group algebra over $k$, $k[G]$. Now, the action of $A[G]$ on $M_x$ given by $(ag)m = a\nabla_g(m)$ defines $M_x$ as an $A[G]$-module because

$$\begin{aligned}((a_1g_1)(a_2g_2))m &= (a_1\nabla_{g_1}(a_2)g_1g_2)m = a_1\nabla_{g_1}(a_2)\nabla_{g_1g_2}(m)\\ &= a_1\nabla_{g_1}(a_2\nabla_{g_2}(m)) = a_1g_1((a_2g_2)m).\end{aligned}$$

Thus the classification of orbits is the classification of the corresponding $A[G]$-modules $M_x$. The main issue of this section is the following definition and the lemma proved by the argument above:

**Definition 2.1.** *An A–G-module is an A module with a G-action such that the two actions commute, that is*

$$\nabla_g(am) = \nabla_g(a)\nabla_g(m).$$

**Lemma 2.1.** *The category of A–G-modules and the category of $A[G]$-modules are equivalent.*

## 3. DEFORMATION THEORY

For $A$ a not necessarily commutative $k$-algebra, $V = \{V_i\}_{i=1}^r$ a swarm of right $A$-modules (which means that $\dim_k \operatorname{Ext}^1_A(V_i,V_j) < \infty$ for $1 \le i,j \le r$), there exists a well-known deformation theory, see [3]. Let $a_r$ be the category of $r$-pointed artinian $k$-algebras. It consists of the commutative diagrams

$$\begin{tikzcd} k^r \arrow[r] \arrow[dr, "\mathrm{Id}"'] & R \arrow[d, "\rho"] \\ & k^r \end{tikzcd}$$

such that $\operatorname{rad}(R) = \ker(\rho)$ fulfills $\operatorname{rad}(R)^n = 0$ for some $n$. Generalizing the commutative case, we set $\hat{a}_r$ equal to the category of complete $r$-pointed $k$-algebras $\hat{R}$ such that $\hat{R}/\operatorname{rad}(\hat{R})^n$ is in $a_r$ for all $n$. Letting $R_{ij} = e_iRe_j$, it is easy to see that $R$ is isomorphic to the matrix algebra $(R_{ij})$. The noncommutative deformation functor $\operatorname{Def}_V : a_r \to \text{Sets}$ is given by

$$\operatorname{Def}_V(R) = \{R \otimes_k A^{op}\text{-modules } V_R | V_R \cong_R (R_{ij} \otimes_k V_j),\, k_i \otimes_R V_R \cong V_i\}/\cong.$$

Let $V_R \in \operatorname{Def}_V(R)$. The left $R$-module structure is the trivial one, and the right $A$-module structure is given by the morphisms $\sigma^R_a : V_i \to R_{ij} \otimes_k V_j$. As in the commutative case, an ($r$-pointed) morphism $\phi : S \twoheadrightarrow R$ is *small* if $\ker\phi \cdot \operatorname{rad}(S) = \operatorname{rad}(S) \cdot \ker\phi = 0$, and for such morphisms, lifting the $\sigma^R$ directly to $S$, the associativity condition gives the obstruction class $o(\phi, V_R) = (\sigma^S_{ab} - \sigma^S_a\sigma^S_b) \in I \otimes_k \operatorname{HH}^2(A, \operatorname{Hom}_k(V_i,V_j))$ where $I = (I_{ij}) = \ker\phi$, such that $V_R$ can be lifted to $V_S$ if and only if $o(V_R,\phi) = 0$, see [3] or [1] for details and complete proofs. Obviously, computations are much easier if $A$ is a commutative $k$-algebra. This is possible to achieve when working with $G$-actions and orbit spaces. For a family $V = \{V_i\}_{i=1}^r$ of $A$–$G$-modules, we put

$$\operatorname{Def}^G_V(R) = \{V_R \in \operatorname{Def}_V(R) | \exists A-G\text{-structure } \nabla : G \to \operatorname{End}(V_R)\} \subseteq \operatorname{Def}_V(R).$$

In [2,3] Laudal constructs the local formal moduli of $A$-modules. In [5,6] applications in the commutative case are given, and in [7] an easy noncommutative example is worked through. In these cites we start with the $k$-algebra $k[\varepsilon] = k[\varepsilon]/\varepsilon^2$ and use the tangent space

$$\operatorname{Def}_V(k[\varepsilon]) \cong (\operatorname{HH}^1(A, \operatorname{Hom}_k(V_i,V_j))) \cong \operatorname{Ext}^1_A(M,M)$$

as dual basis for the local formal moduli $\hat{H}$. The relations among the base elements are given by the obstruction space

$$\operatorname{HH}^2(A, \operatorname{Hom}_k(V_i,V_j)) \cong (\operatorname{Ext}^2_A(V_i,V_j)).$$

## 4. GENERALIZED MATRIX MASSEY PRODUCTS (GMMP)

Let $\{V_i\}_{i=1}^r$ be a given swarm of $A$-modules. For each $i$, choose free resolutions $0 \leftarrow V_i \overset{d_{i,0}}{\leftarrow} L_{i,0} \overset{d_{i,1}}{\leftarrow} L_{i,1} \overset{d_{i,2}}{\leftarrow} L_{i,2} \leftarrow \cdots$. We write

$$L. = \begin{pmatrix} L_{1,.} & 0 & \cdots & 0 \\ 0 & L_{2,.} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & L_{r,.} \end{pmatrix}$$

and we can prove Lemma 4.1 following the proof in [6] step by step:

**Lemma 4.1.** *Let $V_S \in \mathrm{Def}_V(S)$ and let $\phi : R \to S$ be a small surjection. Then there exists a resolution $L^S_. = (S \otimes_k L., d^S_.)$ lifting the complex $L.$, and to give a lifting $V_R$ of $V_S$ is equivalent to lift the complex $L^S_.$ to $L^R_.$.*

*Proof.* Generalized from the commutative case, $M_R \cong_R (R_{ij} \otimes_k M_j)$ is equivalent with $M_R$ $R$-flat. Using this, and tensoring the sequence $0 \to I \to R \to S \to 0$ with $M_R$ over $R$, gives the sequence $0 \to I \otimes_k M \to M_R \to M_S \to 0$. Ordinary diagram chasing then proves that the resolution of $M_S$ can be lifted to an $R$-complex $L.^R$ given the resolution $L.^S$ of $M_S$. Conversely, given a lifting $L.^R$ of the complex $L.^S$ of $M_S$, the long exact sequence proves that this complex is a resolution, and that $M_R = H^0(L.^R)$ is a lifting of $M_S$. □

If $M$ is an $A$–$G$-module where $G$ acts rationally on $A$ and $M$ is a rational $G$-module, finitely generated as an $A$-module, then an $A$-free (projective) resolution of $M$ can be lifted to an $A$–$G$-free resolution, that is a commutative diagram

$$\begin{tikzcd}
0 & V \arrow[l] \arrow[d, "\nabla_g"] & A^{n_0} \arrow[l] \arrow[d, "\nabla_{g,0}"] & A^{n_1} \arrow[l] \arrow[d, "\nabla_{g,1}"] & A^{n_2} \arrow[l] \arrow[d, "\nabla_{g,2}"] & \cdots \arrow[l] \\
0 & V \arrow[l] & A^{n_0} \arrow[l] & A^{n_1} \arrow[l] & A^{n_2} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

This proves that Lemma 4.1 is a particular case of the same lemma with $\mathrm{Def}_V(S)$ replaced by $\mathrm{Def}_V^G(S)$. In [7] we give the definition of GMMP. The tangent space of the deformation functor is $\mathrm{Def}_V^G(E) \cong (\mathrm{Ext}^1_{A-G}(V_i,V_j))$, where $E$ is the noncommutative ring of dual numbers, i.e. $E = k < t_{ij} > /(t_{ij})^2$. For computations we note that when $G$ is reductive and finite dimensional, $\mathrm{Hom}_{A-G}(V_i,V_j) \cong \mathrm{Hom}_A(V_i,V_j)^G$ and $\mathrm{Ext}^1_{A-G}(V_i,V_j) \cong \mathrm{Ext}^1_A(V_i,V_j)^G$, $G$ acting by conjugation. Given a small surjection $\phi : R \to S$, with kernel $I = (I_{ij})$, lift $d.^S$ on $S \otimes_k L.$ to $d.^R$ on $R \otimes_k L.$ in the obvious way. Then $o(\phi, V_S) = \{d_i^R d_{i-1}^R\}_{i \geq 1} \in (I_{ij} \otimes_k \mathrm{Ext}^2_{A-G}(V_i,V_j))$. By the definition of GMMP in [7], these can be read out of the coefficients of a basis in the obstruction space above.

## 5. THE MCKAY CORRESPONDENCE

Let

$$G = \mathbb{Z}_2 =< \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} >=< \tau >$$

act on $\mathbb{A}^2_{\mathbb{C}}$ by $\tau(a,b) = (-a,-b)$. Our goal is to classify the $G$-orbits, and to find a compactification $\tilde{\mathbb{M}}_G \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$ of the orbit space $\mathbb{M}_G$. The existing partial solution is

$$\mathbb{M}_G = \mathrm{Spec}(k[x^2,xy,y^2]) = \mathrm{Spec}(A^G),\, A = k[x,y].$$

This is an orbit space, but not moduli. Consider the point $P=(a,b)=(\sqrt{w},t\sqrt{w})$, $w\neq 0$. Then

$$o(P)=\{(\sqrt{w},t\sqrt{w}),(-\sqrt{w},-t\sqrt{w})\}=Z(I_t),$$

where $I_t=(x^2-w,y-tx)$. We compute the local formal moduli of the $A$–$G$-module $M_t=A/I_t$ from the diagram

$$\begin{tikzcd}
0 & A/I_t \arrow[l] & A \arrow[l] & A^{n_1} \arrow[l] \arrow[d, "\phi"] & A^{n_2} \arrow[l] \arrow[dl, "\equiv 0"] & \cdots \arrow[l] \\
 & & & A/I_t & &
\end{tikzcd}$$

where the upper row is a resolution, we see that in general, $\mathrm{Ext}^1_A(M_t,M_t)\cong \mathrm{Hom}_A(I_t/I_t^2,A/I_t)$ with the action of $G$ given by conjugation, that is the composition given in the sequence

$$I_t \xrightarrow{\nabla_g} I_t \xrightarrow{\phi} A/I_t \xrightarrow{\nabla_{g^{-1}}} A/I_t\ .$$

We get

$$(x^2-w,y-tx) \xrightarrow{\nabla_g} (x^2-w,y-tx) \xrightarrow{\phi} k[x,y]/I_t \xrightarrow{\nabla_{g^{-1}}} k[x,y]/I_t$$

so that $\phi=(\alpha,\beta x)=\alpha(1,0)+\beta(0,x)$ is invariant under the action of $G$. Writing this up in complex form, we get

$$\begin{tikzcd}
0 & M_t \arrow[l] & A \arrow[l] & A^2 \arrow[l, "d_0"'] \arrow[dl, "\xi_1^1"', "\xi_2^1"] & A \arrow[l, "d_1"'] \arrow[dl, "\xi_1^2"', "\xi_2^2"] & 0 \arrow[l] \\
0 & M_t \arrow[l] & A \arrow[l, "d_0"] & A^2 \arrow[l] & A \arrow[l, "d_1"] & 0 \arrow[l]
\end{tikzcd}$$

$$d_0=(x^2-w\ y-tx),\ d_1=\begin{pmatrix}y-tx\\ w-x^2\end{pmatrix},\ \xi_1^1=(1\ 0),\ \xi_2^1=(0\ x),\ \xi_1^2=\begin{pmatrix}0\\-1\end{pmatrix},\ \xi_2^2=\begin{pmatrix}x\\0\end{pmatrix}.$$

We find $\xi_1^1\xi_1^2=\xi_2^1\xi_2^2=\xi_1^1\xi_2^2+\xi_2^1\xi_1^2=0$, which means that all cup-products are identically zero. Thus $\hat{H}_{M_t}=k[[t_1,t_2]]$ with algebraization $H_{M_t}=k[t_1,t_2]$. Because the particular point $\underline{0}=(0,0)$ corresponds to $M_{\underline{0}}=k[x,y]/(x,y)$ with $\mathrm{Ext}^1_{A-G}(M_{\underline{0}},M_{\underline{0}})=0$, we understand that $M_{\underline{0}}$ is a singular point, so that the modulus is $\mathbb{M}_G=(\mathbb{A}^2-\{\underline{0}\})\cup\{\mathrm{pt}\}$. At least in this case, resolving the singularity is a process of compactifying. Given a family $V=\{V_i\}_{i=1}^r$ of simple $A$-modules, an $A$-module $E$ with composition series $E=E_0\supset E_1\supset\cdots\supset E_i\supset E_{i-1}\supset\cdots\supset E_r\supset 0$, where $E_k/E_{k-1}=V_{i_k}$, is called an iterated extension of the family $V$, and the graph $\Gamma(E)$ of $E$ (the representation type) is the graph with nodes in correspondence with $V$ and arrows $\rho_{i_p,i_{p+1}}$ connecting the nodes $V_{i_p}$ and $V_{i_{p+1}}$, identifying arrows if the corresponding extensions are equivalent. In [3] Laudal solves the problem of classifying all indecomposable modules $E$ with fixed extension graph $\Gamma$. He proves that for every $E$ there exists a morphism $\phi: H(V)\to k[\Gamma]$ such that $E\cong\tilde{M}\otimes_\phi k[\Gamma]$, where $\tilde{M}$ is the versal family, resulting in a noncommutative scheme $\mathrm{Ind}(\Gamma)$. In [4], he then proves that the set $\mathrm{Simp}_n(A)$ of $n$-dimensional simple representations of $A$ with the Jacobson topology has a natural scheme structure. He also proves that when $\Gamma$ is a representation graph of dimension $n=\sum_{V\in\Gamma}\dim_k V$, then the set $\mathrm{Simp}(\Gamma)=\mathrm{Simp}_n(A)\cup\mathrm{Ind}(\Gamma)$ has a natural scheme structure with the Jacobson topology, which is a compactification of $\mathrm{Simp}_n(A)$. In our present example, we let $\Gamma$ be the representation type of the regular representation $k[G]$. We construct the composition series $k[G]\cong k[\tau]/(\tau^2-1)\supset(\tau-1)/(\tau^2-1)\supset 0$. Thus we get $V_0=k[\tau]/(\tau-1)\cong k$, $V_1=(\tau-1)/(\tau^2-1)\cong k$ and the action $\nabla^i_\tau$ of $\tau$ on $V_i$ is given by $\nabla^i_\tau=(-1)^i$. From the sequence $(x,y)\xrightarrow{\nabla_\tau}(x,y)\xrightarrow{\phi}V_i\xrightarrow{\nabla_{\tau^{-1}}}V_i$

we immediately see that $\mathrm{Ext}^1_{A-G}(V_i,V_j) = \alpha(1,0)+\beta(0,1)$ when $i \neq j$, $0$ if $i = j$. Writing up the corresponding diagram and multiplying as in the previous example, we get

$$H(V_1,V_2) = \frac{\begin{pmatrix} k & < t_{12}(1), t_{12}(2) > \\ < t_{21}(1), t_{21}(2) > & k \end{pmatrix}}{\begin{pmatrix} t_{12}(1)t_{21}(2) - t_{12}(2)t_{21}(1) & 0 \\ 0 & t_{21}(1)t_{12}(2) - t_{21}(2)t_{12}(1) \end{pmatrix}}.$$

The versal family is given as the cokernel of the morphism

$$\psi : \begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix} \to \begin{pmatrix} H_{11}\otimes A & H_{12}\otimes A \\ H_{21}\otimes A & H_{22}\otimes A \end{pmatrix},$$

$$\psi = \begin{pmatrix} 1\otimes(x,y) & t_{12}(1)\otimes(1,0)+t_{12}(2)\otimes(0,1) \\ t_{21}(1)\otimes(1,0)+t_{21}(2)\otimes(0,1) & 1\otimes(x,y) \end{pmatrix}.$$

Now, as $k[\Gamma] = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, $\phi : H \to k[\Gamma]$ sends both $t_{21}(1)$ and $t_{21}(2)$ to 0. The isomorphism classes of indecomposable $A[G]$-modules with representation type $\Gamma$ are thus given by

$$V_t = \begin{pmatrix} x & y & 0 & 0 \\ -1 & -t & x & y \end{pmatrix}, V_\infty = \begin{pmatrix} x & y & 0 & 0 \\ 0 & -1 & x & y \end{pmatrix}.$$

The inherited group action is $\nabla_\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $k^2$. To find $\mathrm{Simp}(\Gamma)$, we start by computing the local formal moduli of the (worst) module $V_t$, following the algorithm in [2]. We find

$$\mathrm{Ext}^1_{A-G}(V_t,V_t) = \mathrm{Der}_k(A,\mathrm{End}_k(V_t))/\mathrm{Triv} = \left\{ \delta \,|\, \delta(x) = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, \delta(y) = \begin{pmatrix} 0 & w(t+v) \\ v & 0 \end{pmatrix} \right\}$$

by using (in particular) the fact that $xy = yx$ in $A$. Then $H(V_t)^{\mathrm{com}} = k[v,w]$ with versal family $\begin{pmatrix} x & y & -w & -w(t+v) \\ 1 & -(t+v) & x & y \end{pmatrix}$, computed by again using the fact that $xy = yx$ in $A$. While $w = 0$ gives the indecomposable module $V_{v+t}$, $w \neq 0$ gives a simple two-dimensional $A$–$G$-module given by $x^2 = w$, $xy = (t+v)w$, $y^2 = (t+v)^2 w$. This gives an embedding $A^G = k[s_0,s_1,s_2]/(s_0 s_1 - s_2^2) = k[x^2,xy,y^2] \hookrightarrow k[v,w]$ inducing the morphism $\mathrm{Simp}_\Gamma \to \mathrm{Spec}(A_G)$ which is the ordinary blowup of the singular point. The exceptional fibre is $\begin{pmatrix} x & y & 0 & 0 \\ -1 & -t & x & y \end{pmatrix} \cup V_\infty \cong \mathbb{P}^1$.

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**Rühmatoimed, orbiitruumid ja mittekommutatiivne deformatsiooniteooria**

Arvid Siqveland

On vaadeldud rühma $G$ toimet suvalisel $k$-muutkonnal $X = \mathrm{Spec}(A)$. Töös on defineeritud $A$–$G$-mooduleid ja nende deformatsiooniteooriat. On tõestatud, et see deformatsiooniteooria on ekvivalentne moodulite deformatsiooniteooriaga üle mittekommutatiivse $k$-algebra $A[G] = A\sharp G$. Orbiitide klassifikatsiooni võib siis uurida üle kommutatiivse ringi ja töös on antud selle klassifikatsioon tsükliliste singulaarsuste muutkonnal.