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MATHEMATICS

Deformations of embedded Einstein spaces

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Abstract. Many important Einsteinian space-times can be globally embedded into a pseudo-Euclidean flat space of higher dimension N. In this paper we analyse in detail the geometrical properties of infinitesimal deformations of embedded Einstein spaces. Embeddings are defined by N functions $z^A(x^\mu), A = 1,2,\ldots,N, \mu = 0,1,2,3$. Their infinitesimal deformations can be developed in a power series of small parameter ε as follows: $z^A \to \tilde{z}^A = z^A + \varepsilon\, v^A + \varepsilon^2\, w^A + \ldots$. All geometrical quantities can be then expressed in terms of embedding functions z^A and their deformations v^A, w^A, etc. Then we require the deformations to keep Einstein equations satisfied up to a given order in ε. This method can be used to construct approximate solutions of Einstein's equations, and was first introduced in 1978 by one of the authors (RK).

Key words: embeddings, Einstein spaces, gravitational waves.

1. ISOMETRIC EMBEDDINGS AND THEIR PROPERTIES

Embedding of physical space-time into higher dimensional manifolds is a technique used many times while considering cosmological solutions or Kaluza–Klein models; to cite a few, we can mention [1–5]; here our aim is to show that the embedding deformation technique can lead naturally to wave-like solutions describing the propagation of gravitational waves.

Consider the embedding of a four-dimensional Riemannian space parametrized by local coordinates (denoted by x^μ, $\mu,\nu = 0,1,2,3$) in a pseudo-Euclidean space E^N of dimension N. The dimension N, yet unspecified, depends on the topology of the Riemannian space under consideration, and may be quite high, as acknowledged in [6]. Locally, any n-dimensional Riemannian manifold can be embedded in a (pseudo)-Euclidean space of dimension $N = n(n+1)/2$.

Here we are interested in *global* embeddings, which may require a relatively low dimension of the "host" space if the Riemannian space to be embedded possesses some particular symmetry. For example, the de Sitter space can be embedded globally in a five-dimensional pseudo-Euclidean space with signature $(+----)$, and exterior or interior Schwarzschild solutions can be embedded in a six-dimensional E^N with signatures $(++----)$ or $(+-----)$. Consider a global embedding of a Riemannian space V_4 given by the set of *embedding functions* $z^A = z^A\,(x^\mu)$, with $A,B,\ldots = 1,2,\ldots,N,\ \mu,\nu = 0,1,2,3$.

The metric tensor of V_4 is the *induced metric* defined as

$$\overset{\circ}{g}_{\mu\nu} = \eta_{AB}\,\partial_\mu z^A\,\partial_\nu z^B. \tag{1}$$

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The inverse metric tensor $\overset{\circ}{g}{}^{\mu\nu}$ cannot be obtained directly from the embedding functions, but should be computed from the covariant components as their inverse matrix. From now on we use the superscript notation in order to make difference between the "basic" induced metric $\overset{\circ}{g}_{\mu\nu}$, which will be considered as a background, and its infinitesimal deformations expanded in terms of an infinitesimal parameter ε as follows:

$$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + \dots \tag{2}$$

induced by the following deformation of the initial embedding functions:

$$z^A(x^\mu) \to z^A(x^\mu) + \varepsilon \overset{1}{v}{}^A(x^\mu) + \varepsilon^2 \overset{2}{v}{}^A(x^\mu) + \dots . \tag{3}$$

When seen from the ambient pseudo-Euclidean space, the new embedded manifold $\tilde{V}_4$ is the result of an infinitesimal deformation of the initial manifold V_4 induced by a vector field in $E^N_{(p,q)}$, as shown in Fig. 1. It is obvious that such a field, which is defined on the embedded submanifold, can be decomposed into its normal part (in the sense of the pseudo-Euclidean metric) and a part tangent to V_4. The last part induces an internal diffeomorphism of V_4 and can be implemented as a local coordinate transformation. Such deformations do not have any physical meaning, but it is not always necessary to consider exclusively the deformations orthogonal to the embedded V_4; sometimes a deformation having non-vanishing parallel and orthogonal parts can have less non-zero components in the ambient space $E^N_{(p,q)}$ than its part orthogonal to the embedded V_4 manifold.

Our first aim is to express all important geometrical quantities, e.g. the connection coefficients and the curvature tensor, in terms of embedding functions z^A and their partial derivatives. Let us start with Christoffel connection

$$\overset{\circ}{\Gamma}{}^\lambda_{\mu\nu} = \frac{1}{2} \overset{\circ}{g}{}^{\lambda\rho} \left(\partial_\mu \overset{\circ}{g}_{\nu\rho} + \partial_\nu \overset{\circ}{g}_{\mu\rho} - \partial_\rho \overset{\circ}{g}_{\mu\nu} \right). \tag{4}$$

From the definition of $\overset{\circ}{g}_{\mu\nu}$ (1) we have the expression for its partial derivatives:

$$\partial_\lambda \overset{\circ}{g}_{\mu\nu} = \eta_{AB} \left(\partial^2_{\lambda\mu} z^A \, \partial_\nu z^B + \partial_\mu z^A \, \partial^2_{\lambda\nu} z^B \right). \tag{5}$$

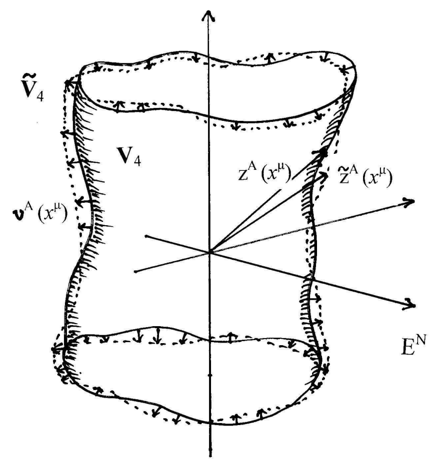

Fig. 1. The initial embedding of a Riemannian manifold V_4 and an infinitesimal deformation producing the new embedding $\tilde{V}_4$.

When substituted into the definition (4) it gives

$$\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} = \eta_{AB}\, \overset{\circ}{g}{}^{\lambda\rho}\, \partial_\rho z^A\, \partial^2_{\mu\nu} z^B. \tag{6}$$

Now, an alternative (although implicit) definition of the Christoffel symbols is contained in the equation that states the vanishing of covariant derivatives of the metric:

$$\nabla_\lambda\, g_{\mu\nu} = 0, \tag{7}$$

which after substitution $\overset{\circ}{g}_{\mu\nu} = \eta_{AB}\, \partial_\mu z^A\, \partial_\nu z^B = \eta_{AB}\, \nabla_\mu z^A\, \nabla_\nu z^B$ leads to the identity

$$\eta_{AB} \left[\nabla_\lambda \nabla_\mu z^A\, \nabla_\nu z^B + \nabla_\mu z^A\, \nabla_\lambda \nabla_\nu z^B \right] = 0. \tag{8}$$

Taking the combination $\nabla_\lambda\, g_{\mu\nu} + \nabla_\mu\, g_{\lambda\nu} - \nabla_\nu\, g_{\mu\lambda} = 0$, we get for arbitrary indices μ, ν, λ:

$$\eta_{AB} \left[\nabla_\lambda \nabla_\mu z^A\, \nabla_\nu z^B \right] = 0. \tag{9}$$

It is interesting to notice that this formula could be derived from the Bianchi identities. This was to be expected as the Bianchi identities are satisfied if the Christoffel symbols are compatible with the metric, which is exactly the meaning of (7). Using this result, let us form the following combination of covariant derivatives which vanishes identically:

$$\eta_{AB} \left[\nabla_\mu (\nabla_\rho\, z^A\, \nabla_\nu \nabla_\sigma\, z^B) - \nabla_\nu (\nabla_\rho\, z^A\, \nabla_\mu \nabla_\sigma\, z^B) \right] = 0. \tag{10}$$

Applying the derivation and using the Leibniz rule we get:

$$\eta_{AB} \left[\nabla_\mu \nabla_\rho z^A\, \nabla_\nu \nabla_\sigma z^B - \nabla_\nu \nabla_\rho z^A\, \nabla_\mu \nabla_\sigma z^B \right] + \eta_{AB} \nabla_\rho\, z^A \left[\nabla_\mu \nabla_\nu \nabla_\sigma\, z^B - \nabla_\nu \nabla_\mu \nabla_\sigma\, z^B \right] = 0.$$

Recalling that

$$\left[\nabla_\mu\, \nabla_\nu - \nabla_\nu\, \nabla_\mu \right] \nabla_\rho\, z^B = \overset{\circ}{R}{}_{\mu\nu}{}^{\lambda}{}_{\rho}\, \nabla_\lambda\, z^B, \tag{11}$$

we can write it:

$$\overset{\circ}{R}_{\mu\nu\lambda\rho} = -\eta_{AB} \left[\nabla_\mu \nabla_\lambda\, z^A \nabla_\nu \nabla_\rho\, z^B - \nabla_\nu \nabla_\lambda\, z^A \nabla_\mu \nabla_\rho\, z^B \right], \tag{12}$$

which is the well-known Gauss–Codazzi equation.

The definition of the Riemann tensor by means of derivatives of the embedding functions given by formula (12) looks compact, but is in fact highly non-linear and complicated. This is so because it contains many Christoffel symbols involved in the second covariant derivatives, which contain the contravariant metric tensor $\overset{\circ}{g}{}^{\mu\nu}$. The components of the contravariant metric tensor are obtained as rational expressions in third and fourth powers of $\nabla_\mu\, z^A$. Nevertheless, the most important point here is that the Riemann tensor depends only on first and second derivatives of embedding functions, so that the Einstein equations expressed in terms of the embedding functions will lead to second-order partial differential equations.

The expressions derived in this section will be very useful in the development of a power series expansion of infinitesimally deformed embedding.

2. INFINITESIMAL DEFORMATIONS OF EMBEDDINGS

Let us consider an isometric embedding of an Einsteinian manifold $\overset{o}{V}_4$ in a pseudo-Euclidean space $E^N_{p,q}$ with signature $(p+,q-)$, with $p+q=N$:

$$\begin{array}{ccc} \overset{o}{V}_4 & \rightarrow & E^N_{p,q} \\ z^A & = & z^A(x^\mu) \end{array} \quad \text{with} \quad \begin{array}{ccc} A,B,\ldots & = & 1,2,\ldots,N, \\ \mu,\nu & = & 0,1,2,3. \end{array} \tag{13}$$

Consider now an infinitesimal deformation of the embedding defined by a converging series of terms proportional to the consecutive powers of a small parameter ε. The deformed embedding defines an Einsteinian space $\tilde{V}_4$:

$$z^A(x^\mu) \rightarrow \tilde{z}^A(x^\mu) = z^A(x^\mu) + \varepsilon\, v^A(x^\mu) + \varepsilon^2 w^A(x^\mu) + \varepsilon^3 h^A(x^\mu) + \ldots . \tag{14}$$

The induced metric on $\tilde{V}_4$ can also be developed in a series of powers of ε:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \overset{o}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + \ldots \\ &= \eta_{AB}\Big[\partial_\mu z^A \partial_\nu z^B + \varepsilon\left(\partial_\mu z^A \partial_\nu v^B + \partial_\mu v^A \partial_\nu z^B\right) + \varepsilon^2\left(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B\right)\Big]. \end{aligned} \tag{15}$$

Among possible infinitesimal deformations of the embedding functions $z^A(x^\mu) + \varepsilon\, v^A(x^\mu)$ there is a large class of functions $v^A(x^\mu)$ which will not alter the intrinsic geometry of the embedded manifold. The translations $v^A = \text{const.}$ obviously do not change the internal metric $\overset{o}{g}_{\mu\nu} = \eta_{AB}\,\partial_\mu z^A \partial_\nu z^B$. Also the generalized Lorentz transformations of the pseudo-Euclidean space $E^N_{(p,q)}$ keep the internal metric unchanged. Indeed, if we set

$$z^A \rightarrow \tilde{z}^A = z^A + \varepsilon\, \Lambda^A_B z^B, \tag{16}$$

with Λ^A_B constant matrix, then the first-order deformed metric is:

$$\tilde{g}_{\mu\nu} = \overset{o}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \ldots = \overset{o}{g}_{\mu\nu} + \varepsilon\left[\eta_{AB}\Lambda^B_C + \eta_{CB}\Lambda^C_A\right]\partial_\mu z^A \partial_\nu z^B. \tag{17}$$

The first-order correction vanishes if the matrices Λ^A_B satisfy the identity

$$\eta_{AB}\Lambda^B_C + \eta_{CB}\Lambda^C_A = 0,$$

which defines infinitesimal rigid rotations (Lorentz transformations) of the pseudo-Euclidean space $E^N_{(p,q)}$.

The geometric character of our approach enables us to eliminate unphysical degrees of freedom using simple geometrical arguments. Remember that in the traditional approach leading to linearized equations for gravitational fields the starting point is the following development of the metric tensor:

$$g_{\mu\nu} = \overset{o}{g}_{\mu\nu} + \varepsilon\, h_{\mu\nu}, \tag{18}$$

thus introducing *ten* components of $h_{\mu\nu}$ as dynamical fields. We know however that most of them do not represent real dynamical degrees of freedom due to the gauge invariance. The metric tensor itself does not correspond to any directly measurable quantity. In fact, its components may be changed by a gauge transformation without changing the components of the Riemann tensor which is the source of measurable gravitational effects. In particular, the gauge transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = g_{\mu\nu} + \nabla_\mu\, \xi_\nu + \nabla_\nu\, \xi_\mu \tag{19}$$

does not alter the Riemann tensor so that both $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ describe the same gravitational field. The arbitrary vector field ξ^μ generating gauge transformation (19) represents four degrees of freedom which are redundant in $g_{\mu\nu}$; this is why in the linearized Einstein equations one may impose four gauge conditions, e.g.

$$\nabla_\mu h^{\mu\nu} = 0. \tag{20}$$

The unphysical degrees of freedom can be easily eliminated from the embedding deformation functions $v^A(x^\mu)$ if we note that any vector field in the embedding space E^N that is *tangent* to the embedded Riemannian space V_4 describes nothing else but a diffeomorphism of V_4, in other words a coordinate change, which has no influence on any physical or intrinsic geometrical quantities.

Vector fields tangent to the four-dimensional embedded manifold V_4 can be decomposed along four arbitrarily chosen independent smooth vector fields in E^N tangent to V_4. On the other hand, vector fields *transversal* to the embedded hypersurface V_4 must satisfy the following obvious *orthogonality conditions*:

$$\eta_{AB}\,\partial_\mu z^A v^B = \eta_{AB}\, v^A \nabla_\mu z^B = 0. \tag{21}$$

For any value of A the four partial derivatives (let us remind that $\nabla_\mu z^A = \partial_\mu z^A$) span a basis of four vector fields in E^N tangent to the submanifold V_4; therefore any vector v^B satisfying the orthogonality condition (21) is transversal to V_4 (as seen in E^N).

The orthogonality condition (21) imposes four independent equations, which reduce the number of independent deformation functions v^A to $N-4$. This means that general non-redundant deformations can be decomposed along $N-4$ independent fields $X^A_{(k)}$, $k = 1,2,\ldots,N-4$:

$$v^A(x^\mu) = \sum_{k=1}^{N-4} v^k(x^\nu)\, X^A_{(k)}(x^\lambda). \tag{22}$$

The basic fields $X^A_{(k)}(x^\lambda)$ can be chosen at will provided they induce a non-singular global vector field on V_4, while the relevant degrees of freedom are contained in $N-4$ functions $v^k(x^\lambda)$. To give an example, the de Sitter space can be globally embedded in a five-dimensional pseudo-Euclidean space $E^5_{1,4}$ with signature $(+----)$; therefore its global deformations can be described by a single function $v(x^\lambda)$ (see [1]).

If the deformation destroys the initial symmetry of the embedded manifold taken as a starting point, the deformed manifold cannot be embedded in the initially sufficient N-dimensional pseudo-Euclidean space but might need a flat embedding space of higher dimension.

As long as we investigate only the first-order corrections to geometry, we should not worry about this issue for the following two reasons: first, when a global embedding is given, its infinitesimal deformations cannot lead to a global modification of topology; second, if the larger embedding space was introduced, say E^{N+m}, it would contain the initial embedding space E^N as its linear subspace, so that $E^{N+m} = E^N \oplus E^m$, and its pseudo-Euclidean metric could be represented as a blockwise reducible matrix

$$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & \eta_{ij} \end{pmatrix}, \tag{23}$$

with $A,B,\ldots = 1,2,\ldots,N$, $i,j,\ldots = 1,2,\ldots,m$, $\alpha,\beta,\ldots = 1,2,\ldots,N+m$. Accordingly, any deformation of the initial embedding can be decomposed in two parts, one contained in the initial embedding space E^N and the other in the complementary subspace E^m:

$$v^\alpha = [v^A, v^m]. \tag{24}$$

But the initial embedding functions had their components entirely in the first subspace E^N, $z^\alpha = [z^A, 0]$, therefore the deformed embedding functions can be written as

$$\tilde{z}^\alpha = [z^A + \varepsilon v^A, \varepsilon v^k], \tag{25}$$

so that the induced metric of the deformed embedding will be

$$\begin{aligned} g_{\mu\nu} &= \overset{\mathrm{o}}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + \ldots \\ &= \eta_{AB} \left(\partial_\mu z^A \partial_\nu z^B\right) + \varepsilon\, \eta_{AB} \left(\partial_\mu z^A \partial_\nu v^B + \partial_\mu v^A \partial_\nu z^B\right) \\ &\quad + \varepsilon^2 \left[\eta_{AB} \left(\partial_\mu v^A \partial_\nu v^B + \partial_\mu z^A \partial_\nu w^B + \partial_\mu w^A \partial_\nu z^B\right) + \eta_{ij} \left(\partial_\mu v^i \partial_\nu v^j\right)\right] \ldots . \end{aligned} \tag{26}$$

From this one can see that the deformations towards extra dimensions do not contribute to the first-order corrections of any geometrical quantities obtained from the deformed embedding functions. This is why we shall not consider such deformations while investigating at first only the terms linear in the infinitesimal parameter ε.

Our principal aim now is to establish the explicit form of connection and curvature components induced on the infinitesimally deformed embedding $\tilde{V}_4$. To this end we must calculate the approximate expression of the contravariant metric tensor $g^{\mu\nu}$. If the covariant metric is decomposed as in (2),

$$g_{\mu\nu} = \overset{\mathrm{o}}{g}_{\mu\nu} + \varepsilon \overset{1}{g}_{\mu\nu} + \varepsilon^2 \overset{2}{g}_{\mu\nu} + \ldots,$$

then we have the following formulae defining the corresponding decomposition of $g^{\mu\nu}$:

$$\begin{aligned} g^{\mu\nu} &= \overset{\mathrm{o}}{g}{}^{\mu\nu} + \varepsilon \overset{1}{g}{}^{\mu\nu} + \varepsilon^2 \overset{2}{g}{}^{\mu\nu} + \ldots \\ &= g^{\mu\nu} - \varepsilon \overset{\mathrm{o}}{g}{}^{\mu\rho} \overset{\mathrm{o}}{g}{}^{\nu\sigma} \overset{1}{g}_{\rho\sigma} - \varepsilon^2 \left[\overset{\mathrm{o}}{g}{}^{\mu\rho} \overset{\mathrm{o}}{g}{}^{\nu\sigma} \overset{2}{g}_{\rho\sigma}\right] + \varepsilon^2 \left[\overset{\mathrm{o}}{g}{}^{\mu\rho} \overset{\mathrm{o}}{g}{}^{\nu\sigma} \overset{\mathrm{o}}{g}{}^{\lambda\kappa} \overset{1}{g}_{\rho\lambda} \overset{1}{g}_{\sigma\kappa}\right], \end{aligned} \tag{27}$$

which shows that also the contravariant metric tensor depends exclusively on the first derivatives of the embedding functions, although in a quite complicated manner. In what follows we shall keep only the first-order terms linear in ε.

Let us start by computing the first (linear) correction to the components of the Christoffel connection, which develops in Taylor series as

$$\Gamma^\lambda_{\mu\nu} = \overset{\mathrm{o}}{\Gamma}{}^\lambda_{\mu\nu} + \varepsilon \overset{1}{\Gamma}{}^\lambda_{\mu\nu} + \varepsilon^2 \overset{2}{\Gamma}{}^\lambda_{\mu\nu} + \ldots, \tag{28}$$

then by definition we have:

$$\begin{aligned} \overset{1}{\Gamma}{}^\lambda_{\mu\nu} &= \frac{1}{2} \overset{\mathrm{o}}{g}{}^{\lambda\rho} \left(\partial_\mu \overset{1}{g}_{\nu\rho} + \partial_\nu \overset{1}{g}_{\mu\rho} - \partial_\rho \overset{1}{g}_{\mu\nu}\right) + \frac{1}{2} \overset{1}{g}{}^{\lambda\rho} \left(\partial_\mu \overset{\mathrm{o}}{g}_{\nu\rho} + \partial_\nu \overset{\mathrm{o}}{g}_{\mu\rho} - \partial_\rho \overset{\mathrm{o}}{g}_{\mu\nu}\right) \\ &= \frac{1}{2} \overset{\mathrm{o}}{g}{}^{\lambda\rho} \left(\partial_\mu \overset{1}{g}_{\nu\rho} + \partial_\nu \overset{1}{g}_{\mu\rho} - \partial_\rho \overset{1}{g}_{\mu\nu}\right) - \frac{1}{2} \overset{\mathrm{o}}{g}{}^{\lambda\sigma} \overset{\mathrm{o}}{g}{}^{\rho\kappa} \overset{1}{g}_{\sigma\kappa} \left(\partial_\mu \overset{\mathrm{o}}{g}_{\nu\rho} + \partial_\nu \overset{\mathrm{o}}{g}_{\mu\rho} - \partial_\rho \overset{\mathrm{o}}{g}_{\mu\nu}\right). \end{aligned} \tag{29}$$

This, after some algebra, gives

$$\overset{1}{\Gamma}{}^\lambda_{\mu\nu} = \eta_{AB} \overset{\mathrm{o}}{g}{}^{\lambda\rho} \left[\nabla_\rho z^A \nabla_\mu \nabla_\nu v^B + \nabla_\rho v^A \nabla_\mu \nabla_\nu z^B\right]. \tag{30}$$

This expression has a tensorial character as it should be, because by definition both quantities $\Gamma^\lambda_{\mu\nu}$ and $\tilde{\Gamma}^\lambda_{\mu\nu}$ transform as connection coefficients, therefore their difference must transform as a tensor, and this is true for any term of the development into series of powers of ε.

The coefficients $\overset{1}{\Gamma}{}^\lambda_{\mu\nu}$ will be useful for the derivation of geodesic equations in the deformed space-time, but they are not necessary for the computation of the first-order deformation of the Riemann tensor, which can be determined as follows. Let us develop second covariant derivatives of the deformed embedding functions $\tilde{z}^A(x^\mu)$ in a series of the infinitesimal parameter ε. The first linear term is given by:

$$\begin{aligned}\tilde{\nabla}_\mu \tilde{\nabla}_\nu \tilde{z}^A &= \tilde{\nabla}_\mu \tilde{\nabla}_\nu z^A + \varepsilon \tilde{\nabla}_\mu \tilde{\nabla}_\nu v^A + O(\varepsilon^2) \\ &= \nabla_\mu \nabla_\nu z^A + \varepsilon \left[\nabla_\mu \nabla_\nu v^A - \overset{1}{\Gamma}{}^\lambda_{\mu\nu} \nabla_\lambda z^A\right] + O(\varepsilon^2).\end{aligned} \tag{31}$$

The Riemann tensor induced on the deformed embedding is defined by the same formula as in the previous section (Eq. (12)).

$$\tilde{R}_{\nu\mu\lambda\rho} = \eta_{AB}\left[\tilde{\nabla}_\mu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\nu \tilde{\nabla}_\rho \tilde{z}^B - \tilde{\nabla}_\nu \tilde{\nabla}_\lambda \tilde{z}^A \tilde{\nabla}_\mu \tilde{\nabla}_\rho \tilde{z}^B\right]. \tag{32}$$

Note that in order to calculate the components of the Riemann tensor induced on the deformed manifold $\tilde{V}_4$ we use not only the deformed embedding functions $\tilde{z}^A$, but also the "deformed" covariant derivations $\tilde{\nabla}_\mu$. Now, when we insert the expressions like (31) into the definition of Riemann tensor components (32), we shall encounter, besides the zeroth-order initial Riemann tensor $\overset{\mathrm{o}}{R}_{\mu\nu\lambda\rho}$ and the second-order corrections proportional to ε^2, just two types of terms linear in ε:

$$\varepsilon\, \eta_{AB} \nabla_\mu \nabla_\lambda z^A \nabla_\nu \nabla_\rho v^B$$

and

$$\varepsilon\, \eta_{AB} \nabla_\mu \nabla_\lambda z^A \overset{1}{\Gamma}{}^\lambda_{\nu\rho} \nabla_\lambda z^B. \tag{33}$$

The terms of the second type vanish by virtue of the identity (9); therefore the first-order correction to the components of the Riemann tensor can be written as follows:

$$\begin{aligned}\overset{1}{R}_{\nu\mu\lambda\rho} &= \eta_{AB}\Big[\nabla_\mu \nabla_\lambda z^A \nabla_\nu \nabla_\rho v^B + \nabla_\mu \nabla_\lambda v^A \nabla_\nu \nabla_\rho z^B \\ &\quad - \nabla_\nu \nabla_\lambda z^A \nabla_\mu \nabla_\rho v^B - \nabla_\nu \nabla_\lambda v^A \nabla_\mu \nabla_\rho z^B\Big].\end{aligned} \tag{34}$$

To establish the form of linear correction to Einstein's equations we need to know the components of the first-order correction to the Ricci tensor and the Riemann scalar. These quantities are readily computed as follows:

$$\overset{1}{R}_{\mu\rho} = \overset{\mathrm{o}}{g}{}^{\nu\lambda} \overset{1}{R}_{\mu\nu\lambda\rho} + \overset{1}{g}{}^{\nu\lambda} \overset{\mathrm{o}}{R}_{\mu\nu\lambda\rho}, \quad \overset{1}{R} = \overset{\mathrm{o}}{g}{}^{\mu\nu} \overset{1}{R}_{\mu\nu} + \overset{1}{g}{}^{\mu\nu} \overset{\mathrm{o}}{R}_{\mu\nu}. \tag{35}$$

Finally, the first-order correction to the Einstein tensor, i.e. the left-hand side of Einstein's equations is:

$$\begin{aligned}\overset{1}{G}_{\mu\nu} &= \overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{1}{g}_{\mu\nu}\overset{o}{R} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{1}{R} \\ &= \overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\overset{1}{R}_{\lambda\rho} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{1}{g}{}^{\lambda\rho}\overset{o}{R}_{\lambda\rho} - \frac{1}{2}\overset{1}{g}_{\mu\nu}\overset{o}{R}.\end{aligned} \tag{36}$$

In what follows, we shall always suppose that the initial Riemannian manifold is a solution of Einstein's equations, i.e. an Einstein space which is Ricci-flat and consequently has zero scalar curvature, too. Therefore the linear correction (of the first order in small parameter ε) to the Einstein tensor will reduce to:

$$\overset{1}{R}_{\mu\nu} - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\overset{1}{R}_{\lambda\rho} = \left(\delta^\lambda_\mu\, \delta^\rho_\nu - \frac{1}{2}\overset{\mathrm{o}}{g}_{\mu\nu}\overset{\mathrm{o}}{g}{}^{\lambda\rho}\right)\overset{1}{R}_{\lambda\rho}. \tag{37}$$

In the absence of any extra gravitating matter (besides the matter generating the basic solution, e.g. the central spherical body for Schwarzschild's solution) the expression (37) must be equal to zero. But this amounts to the Ricci flatness up to the first order, because the operator acting on the right on the Ricci tensor in (37) is non-singular (in fact, it is its own inverse).

From this we infer that in an Einsteinian background the first-order correction *in vacuo* should satisfy the equation

$$\overset{1}{R}_{\lambda\rho} = 0. \tag{38}$$

This may be written, developing (35), as:

$$\overset{1}{R}_{\nu\sigma} = U_{\nu\sigma}{}^{\mu\gamma}{}_{A} \nabla_\mu \nabla_\gamma v^A + V_{\nu\sigma}{}^{\mu}{}_{A} \nabla_\mu v^A = 0 \tag{39}$$

with $U_{\nu\sigma}{}^{\mu\gamma}{}_{A}$ and $V_{\nu\sigma}{}^{\mu}{}_{A}$ linear operators containing the embedding functions only. In the case when the energy-momentum tensor is present (supposing however that it describes the influence of matter weak enough in order to keep the basic solution unchanged), one must use the full Einstein's tensor on the right-hand side. The first correction, linear in ε, reduces then to only two terms due to the fact that the initial solution is an Einstein space *in vacuo* so that $\overset{0}{R}_{\lambda\rho} = 0$ and $\overset{0}{R} = 0$:

$$\overset{1}{R}_{\mu\nu} - \frac{1}{2} \overset{\circ}{g}_{\mu\nu} \overset{\circ}{g}{}^{\lambda\rho} \overset{1}{R}_{\lambda\rho} = \left[\delta^\lambda_\mu \, \delta^\rho_\nu - \frac{1}{2} \overset{\circ}{g}_{\mu\nu} \overset{\circ}{g}{}^{\lambda\rho} \right] \overset{1}{R}_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{40}$$

and this in turn, due to the idempotent property of the operator in brackets, can be written equivalently as

$$\overset{1}{R}_{\mu\nu} = -\frac{8\pi G}{c^4} \left[\delta^\lambda_\mu \, \delta^\rho_\nu - \frac{1}{2} \overset{\circ}{g}_{\mu\nu} \overset{\circ}{g}{}^{\lambda\rho} \right] T_{\lambda\rho}, \tag{41}$$

which may prove to be more practical for further calculations especially when the energy-momentum tensor has a particularly simple form.

Let us this time keep the terms up to the third order in the expansion (14).

The metric tensor $\tilde{g}_{\mu\nu}$ of the deformed embedding is given by the formula (15), and the first order correction to the Christoffel connection coefficients was already given in Eq. (30).

The second-order correction to the Christoffel connection can be found from the expansion in powers of ε of the formula (9), which is valid also on the deformed manifold:

$$\eta_{AB} \left[\tilde{\nabla}_\lambda \tilde{\nabla}_\mu \tilde{z}^A \, \tilde{\nabla}_\nu \tilde{z}^B \right] = 0. \tag{42}$$

From the expansion, identifying the terms proportional to ε^2, one readily gets:

$$\overset{2}{\Gamma}{}^{\lambda}_{\mu\nu} = \eta_{AB} \overset{\circ}{g}{}^{\lambda\rho} \left[\nabla_\rho v^A \, \nabla_\mu \nabla_\nu \, v^B + \nabla_\rho w^A \, \nabla_\mu \nabla_\nu \, z^B + \nabla_\rho z^A \, \nabla_\mu \nabla_\nu w^B - \overset{1}{\Gamma}{}^{\sigma}_{\mu\nu} \left(\nabla_\rho \, v^A \nabla_\sigma \, z^B + \nabla_\rho \, z^A \nabla_\sigma \, v^B \right) \right]. \tag{43}$$

The second-order terms in the expansion of the Riemann tensor are given by the following formula:

$$\begin{aligned} \overset{2}{R}_{\mu\nu\sigma\rho} \quad = \quad & \eta_{AB} \Big[\nabla_\nu \nabla_\sigma z^A \left(\nabla_\mu \nabla_\rho w^B - \overset{1}{\Gamma}{}^{\kappa}_{\mu\rho} \nabla_\kappa v^B - \overset{2}{\Gamma}{}^{\kappa}_{\mu\rho} \nabla_\kappa z^B \right) \\ & + \left(\nabla_\nu \nabla_\sigma v^A - \overset{1}{\Gamma}{}^{\kappa}_{\nu\sigma} \nabla_\kappa z^A \right) \left(\nabla_\mu \nabla_\rho v^B - \overset{1}{\Gamma}{}^{\kappa}_{\mu\rho} \nabla_\kappa z^B \right) \\ & + \left(\nabla_\nu \nabla_\sigma w^A - \overset{1}{\Gamma}{}^{\kappa}_{\nu\sigma} \nabla_\kappa v^A - \overset{2}{\Gamma}{}^{\kappa}_{\nu\sigma} \nabla_\kappa z^A \right) \nabla_\mu \nabla_\rho z^B - (\mu \leftrightarrow \nu) \Big]. \end{aligned} \tag{44}$$

The terms proportional to $\overset{2}{\Gamma}$ are zero according to (9), because they contain the products of first and second covariant derivatives of the original embedding functions contracted with the Euclidean metric, $\eta^{AB} \nabla_\mu \nabla_\lambda z^A \nabla_\nu z^B = 0$. With a little algebra we find:

$$\overset{2}{R}_{\mu\nu\sigma\rho} = \eta_{AB}\left[\nabla_\nu\nabla_\sigma z^A\nabla_\mu\nabla_\rho w^B + \nabla_\nu\nabla_\sigma w^A\nabla_\mu\nabla_\rho z^B - \nabla_\mu\nabla_\sigma z^A\nabla_\nu\nabla_\rho w^B - \nabla_\mu\nabla_\sigma w^A\nabla_\nu\nabla_\rho z^B \right.$$
$$\left. + \nabla_\nu\nabla_\sigma v^A\nabla_\mu\nabla_\rho v^B - \nabla_\mu\nabla_\sigma v^A\nabla_\nu\nabla_\rho v^B\right] + \overset{o}{g}_{\kappa\lambda}\left[\overset{1}{\Gamma}{}^\kappa_{\mu\sigma}\overset{1}{\Gamma}{}^\lambda_{\nu\rho} - \overset{1}{\Gamma}{}^\kappa_{\nu\sigma}\overset{1}{\Gamma}{}^\lambda_{\mu\rho}\right],$$

where the $\overset{1}{\Gamma}\overset{1}{\Gamma}$ terms contain only z and v functions. To write down the second-order correction to Einstein equations we need also the explicit form of the correction to the Ricci tensor:

$$\overset{2}{R}_{\nu\rho} = \overset{o}{g}{}^{\mu\sigma}\overset{2}{R}_{\mu\nu\sigma\rho} + \overset{1}{g}{}^{\mu\sigma}\overset{1}{R}_{\mu\nu\sigma\rho} + \overset{2}{g}{}^{\mu\sigma}\overset{o}{R}_{\mu\nu\sigma\rho} \tag{45}$$

and the Riemann scalar

$$\overset{2}{R} = \overset{o}{g}{}^{\mu\sigma}\overset{2}{R}_{\mu\sigma} + \overset{1}{g}{}^{\mu\sigma}\overset{1}{R}_{\mu\sigma} + \overset{2}{g}{}^{\mu\sigma}\overset{o}{R}_{\mu\sigma} \tag{46}$$

with the w-functions contained only in $\overset{2}{R}_{\mu\nu\sigma\rho}$ and in $\overset{2}{g}{}^{\mu\sigma}$.

Looking for vacuum solutions we must develop the full expression for the second-order correction to Einstein's tensor. For the sake of simplicity we note only that the operators acting on the derivative of w^A are the same that appear in Eq. (39). The already known functions z^A and v^A serve now as the right-hand side of the equations determining the w^A functions:

$$\overset{2}{R}_{\nu\rho} = 0 \Longrightarrow U_{\nu\sigma}{}^{\xi\gamma}{}_A\nabla_\xi\nabla_\gamma w^A + V_{\nu\sigma}{}^{\xi}{}_A\nabla_\xi w^A = B_{\nu\rho}(v^A, z^A)$$

with $B_{\nu\rho}(v^A, z^A)$ combination of derivatives of v^A and z^A functions.

Finally, the third-order correction to the Riemann tensor can be obtained by expanding Eq. (32) and taking all the third-order quantites. This gives the following result:

$$\overset{3}{R}_{\mu\nu\sigma\rho} = \eta_{AB}\left[\nabla_\nu\nabla_\sigma z^A\nabla_\mu\nabla_\rho h^B + \nabla_\nu\nabla_\sigma h^A\nabla_\mu\nabla_\rho z^B \right.$$
$$\left. + \nabla_\mu\nabla_\sigma v^A\nabla_\nu\nabla_\rho w^B + \nabla_\mu\nabla_\sigma w^A\nabla_\nu\nabla_\rho v^B - (\mu\leftrightarrow\nu)\right]$$
$$+ \left[\overset{o}{g}_{\kappa\lambda}\left(\overset{2}{\Gamma}{}^\kappa_{\mu\sigma}\overset{1}{\Gamma}{}^\lambda_{\nu\rho} + \overset{1}{\Gamma}{}^\kappa_{\mu\sigma}\overset{2}{\Gamma}{}^\lambda_{\nu\rho}\right) + \overset{1}{g}_{\kappa\lambda}\overset{1}{\Gamma}{}^\kappa_{\mu\sigma}\overset{1}{\Gamma}{}^\lambda_{\nu\rho} - (\mu\leftrightarrow\nu)\right]. \tag{47}$$

This allows us to write down the third-order correction to Einstein's equations, getting the explicit expression for the third-order corrections to the Ricci tensor and the curvature scalar, in the same way followed in Eqs (45) and (46).

3. EXAMPLE OF APPROXIMATE SOLUTION: PLANE GRAVITATIONAL WAVE

The usual approach to the description of gravitational waves in General Relativity consists in using the post-Newtonian approximations. This approach was developed in [7–10] and many other publications. This approach starts with the exact Newtonian solution of the two-body problem, then by adding relativistic corrections arrives at perturbations in the form of gravitational waves. On the other hand, an alternative approach to the description of motions in General Relativity, including the two-body problem, was developed more recently in [11,12], where small deformations of a given exact solution inside General Relativity were used. Finally, the deformation technique used in order to describe radiative phenomena was also used in [13]. Here we shall show how the embedding deformation formalism can naturally lead to wave-like solutions.

In a Minkowskian space-time M_4 parameterized by Cartesian coordinates $x^\mu = [ct, x, y, z]$ all connection coefficients, as well as the components of the Riemann and Ricci tensors, identically vanish. The flat Minkowskian space can be embedded as a hyperplane in any pseudo-Euclidean space with more than four dimensions and signature $(1+, (N-1)-)$. Let us choose the simplest case of embedding in five dimensions:

$$M_4 \to E^5_{1,4}$$

with the first four components denoting a Minkowskian space-time vector in Cartesian coordinates:

$$z^1 = ct, z^2 = x, z^3 = y, z^4 = z, \quad z^5 = 0. \tag{48}$$

The last Cartesian coordinate is considered as an extra dimension of $E^5_{1,4}$ orthogonal to the M_4 hyperplane. All covariant derivatives in (32) can be replaced by partial derivatives, and all second derivatives of linear embedding functions are identically zero. Therefore, in order to investigate non-trivial deformations of the Minkowskian space embedded as a hyperplane we must go the second order in ε. This leads to the following equation resulting from the requirement of the vanishing of the Ricci tensor:

$$\overset{2}{R}_{\mu\rho} = 0 \Longrightarrow \overset{\circ}{g}^{\lambda\nu} \eta_{AB} \left[\nabla_\mu \nabla_\lambda v^A \nabla_\nu \nabla_\rho v^B - \nabla_\nu \nabla_\lambda v^A \nabla_\mu \nabla_\rho v^B\right] = 0.$$

We shall not consider infinitesimal deformations of the first four coordinates because they coincide with coordinate transformations in V_4; therefore the only non-vanishing component of v^A is the remaining fifth coordinate deformation, expanded in a series of powers of ε:

$$z^5 = \varepsilon\, v(x^\mu) + \varepsilon^2\, w(x^\mu) + \varepsilon^3\, h(x^\mu) + \dots .$$

In order to keep the Einstein equations satisfied after deformation up to the second order terms, we must have

$$\overset{2}{R}_{\mu\rho} = \overset{\circ}{g}^{\lambda\nu} \left[\nabla_\mu \nabla_\lambda v \nabla_\nu \nabla_\rho v - \nabla_\nu \nabla_\lambda v \nabla_\mu \nabla_\rho v\right] = 0. \tag{49}$$

Any function of linear combination of Cartesian coordinates is an obvious solution of Eq. (49). Indeed, if we set $v(x^\mu) = f(k_\mu x^\mu)$, inserting the derivatives of $v(k_\mu x^\mu)$ into (49), the following simple equation is obtained:

$$\overset{\circ}{g}^{\lambda\nu} \left[k_\mu k_\lambda\, k_\nu k_\rho\, v'^2 - k_\nu k_\lambda\, k_\mu k_\rho\, v'^2\right] = k_\nu k^\nu\, v'^2 \left[k_\mu k_\rho - k_\rho k_\mu\right] = 0. \tag{50}$$

But in fact this deformation does not have any physical meaning, because the Riemann tensor, which is the only observable quantity, identically vanishes:

$$\overset{2}{R}_{\mu\nu\lambda\rho} = \left[k_\mu\, k_\lambda\, k_\nu\, k_\rho - k_\nu\, k_\lambda\, k_\mu\, k_\rho\right] = 0. \tag{51}$$

The vanishing of the Riemann tensor is not surprising because the deformation considered looks like a deformation of a plane into a cylinder, which does not alter its intrinsic flat geometry.

The fact that there are no wave-like solutions at the first order of deformation of Minkowskian space-time suggests that the same situation will prevail when we investigate other Einsteinian manifolds embedded in a pseudo-Euclidean flat space, e.g. the Schwarzschild solution. If the contrary was true, one could keep the wave-like propagating deformations also in the flat limit, which would contradict the absence of such solutions among the first-order deformations of the Minkowskian space-time.

This means that the only hope to produce contributions to the Riemann tensor behaving like a propagating gravitational field, i.e. the gravitational waves, is to consider the third- (and higher) order deformations of embedded Einsteinian manifolds. The third-order variation for the Riemann tensor in the case of deformations of all orders orthogonal to the embedded manifold is given by:

$$\overset{3}{R}_{\mu\nu\sigma\rho} = \eta_{AB}\left(\nabla_\mu\nabla_\rho v^A\nabla_\nu\nabla_\sigma w^B + \nabla_\mu\nabla_\rho w^A\nabla_\nu\nabla_\sigma v^B - \nabla_\nu\nabla_\rho v^A\nabla_\mu\nabla_\sigma w^B - \nabla_\nu\nabla_\rho w^A\nabla_\mu\nabla_\sigma v^B\right). \quad (52)$$

The linear contribution coming from the expressions containing third-order deviation linearly does vanish because the derivatives of the corresponding z^5 coordinate are identically zero.

A wave-like behaviour of the Riemann tensor can be produced if we assume that w depends on variables orthogonal to the worldlines parallel to the vector k. For the sake of simplicity, let us start with the first-order deformation in the direction of the fifth coordinate, i.e. orthogonal to the embedded Minkowskian hyperplane M_4 as a plane wave propagating along the z-axis:

$$\varepsilon\, v(x^\mu) = A\cos(\omega t - kz).$$

According to our general analysis, by virtue of (51), this deformation does not contribute to the Riemann tensor, which remains zero even at the second order. Now let us add up the second-order deformation depending on the variables x and y only:

$$z^5 = \varepsilon A\cos(\omega t - kz) + \varepsilon^2 w(x,y). \quad (53)$$

The only contribution to the third-order correction to the Riemann tensor has the form given by the formula (52), in which the covariant derivatives can be replaced by partial derivatives given that all Christoffel symbols vanish in Cartesian coordinates. The function $w(x,y)$ must have some non-vanishing second-order derivatives; let us make the simplest choice and set $w(x,y) = Bxy$, with $B = \text{const.}$ having the dimension cm^{-1}.

Then the only non-vanishing second derivative is $\partial^2_{xy} w = B$. Taking into account the form of (52), the only non-vanishing components are:

$$\overset{3}{R}_{\mu xy\rho} = \left(\partial^2_{\mu y} v\,\partial^2_{x\rho} w + \partial^2_{\mu y} w\,\partial^2_{x\rho} v - \partial^2_{xy} v\,\partial^2_{\mu\rho} w - \partial^2_{xy} w\,\partial^2_{\mu\rho} v\right), \quad (54)$$

and all other components obtained from this one by permutations of indexes allowed by the well-known symmetries of the Riemann tensor, like e.g. $\overset{3}{R}_{x\mu\rho y}$, etc.

Now, given that v does not depend on x and on y, the only non-vanishing term in (54) is the one containing $\partial_x\partial_y w = B$; so we have

$$\overset{3}{R}_{\mu xy\rho} = -\partial^2_{xy} w\,\partial^2_{\mu\rho} v = -B\,\partial^2_{\mu\rho} v. \quad (55)$$

There is no contribution to the Ricci tensor coming from $\overset{\circ}{g}{}^{xy}\,\overset{3}{R}_{\mu xy\rho}$ because the Minkowskian metric tensor is diagonal and $\overset{\circ}{g}{}^{xy} = 0$; therefore, to make the Ricci tensor vanish up to the third order means that the following equation must be satisfied:

$$\overset{\circ}{g}{}^{\mu\rho}\,\overset{3}{R}_{\mu xy\rho} = -B\,\overset{\circ}{g}{}^{\mu\rho}\,\partial^2_{\mu\rho} v = 0. \quad (56)$$

This is the wave equation for v, imposing the dispersion relation $\omega^2 = c^2 k^2$.

This particular form of the "modulating" function $w(x,y)$ can be easily generalized. As a first step, let us consider an arbitrary quadratic form in variables x and y: let us put

$$w = Ax^2 + Bxy + Cy^2.$$

Besides the non-vanishing component $\overset{3}{R}_{\mu xy\rho}$, two other components of the Riemann tensor will appear now:

$$\overset{3}{R}_{\mu xx\rho} = 2A$$

and

$$\overset{3}{R}_{\mu yy\rho} = 2C,$$

which have the same structure as the (x,y) component (55):

$$\overset{3}{R}_{\mu xx\rho} = -\partial^2_{xx} w\, \partial^2_{\mu\rho} v = -2A\, \partial^2_{\mu\rho} v,\ \overset{3}{R}_{\mu yy\rho} = -\partial^2_{yy} w\, \partial^2_{\mu\rho} v = -2C\, \partial^2_{\mu\rho} v. \tag{57}$$

The components (xx), (xy), and (yy) of the Ricci tensor vanish if the same condition (56) is satisfied; but now we shall also make sure that all other components of the Ricci tensor vanish, too, which will be true if the following trace is zero:

$$\overset{\mathrm{o}}{g}{}^{xx}\overset{3}{R}_{\mu xx\rho} + \overset{\mathrm{o}}{g}{}^{yy}\overset{3}{R}_{\mu yy\rho} = -\partial^2_{xx} w\, \partial^2_{\mu\rho} v - \partial^2_{xx} w\, \partial_\mu \partial_\rho v = -(2A+2C)\, \partial^2_{\mu\rho} v, \tag{58}$$

leading to the extra condition on the coefficients A and C, namely, $A = -C$, thus leaving only two degrees of freedom for the function w. This suggests the *quadrupolar* character of the gravitational wave, which deforms the space simultaneously in two directions perpendicular to the direction of propagation; notice that if w depended only on one transversal variable, say x, the vanishing of the Ricci tensor would impose $w = 0$ (or a constant, which would not have any physical meaning at all). It is also worthwhile to note that the fact the planar wave solutions appear only at the *third order* of deformation echoes the well-known result obtained via linearization of the metric tensor, telling that gravitational waves are emitted when the *third* time derivative of the quadrupolar moment is different from zero.

The same is true for any homogeneous polynomial of two variables x and y, provided it satisfies the two-dimensional Laplace equation $\partial^2_{xx} w + \partial^2_{yy} w = 0$. Finally, we can generalize our result by stating that the deformation of Minkowskian space-time embedded as a hyperplane in a five-dimensional Euclidean ambient space leads to the vanishing of the Ricci tensor up to the third order in the small parameter ε if it has the form

$$z^5 = \varepsilon \cos(\omega t - kz + \Phi) + \varepsilon^2 w(x,y) + O(\varepsilon^3). \tag{59}$$

Provided that v satisfies $\omega^2 = c^2 k^2$ and w satisfies the two-dimensional Laplace equation $\nabla^2 w = 0$; Φ denotes the arbitrary phase that one can add to the argument of the cosine function.

Taking into account that the corresponding Riemann tensor $\varepsilon^3 \overset{3}{R}_{\mu\nu\lambda\rho}$ is linear both in v and w, we can compose by superposition a transversally polarized plane wave of arbitrary shape and spectrum, propagating with the phase velocity equal to the speed of light.

However, although from a purely mathematical point of view a $w(x,y)$ satisfying the two-dimensional Laplace equation becomes a solution to the third-order correction to Einstein's equations, we claim that only the quadratic functions of x and y represent a physically acceptable case. This is because constant and linear functions w make vanish not only the Ricci tensor, but also the Riemann tensor (due to the second derivatives of the zero-order *linear* embedding functions z^A).

The quadratic functions w lead to a constant (on the x,y-plane) Riemann tensor, multiplied by the oscillating factor $\cos(\omega t - kz)$, which is acceptable for an infinite plane wave.

However, if we take a cubic or higher degree polynomial in x,y as the modulating function w, the Riemann tensor will become linear (or higher degree) in x,y, becoming infinite at spatial infinity, which is physically unacceptable. Therefore only the second degree polynomial fully characterizes the plane gravitational wave, conveying it only two degrees of freedom, and typical for spin 2.

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Sisestatud Einsteini ruumide deformatsioonid

Richard Kerner ja Salvatore Vitale

Mitmed füüsikaliselt olulised Einsteini aegruumid on globaalselt sisestatavad pseudoeukleidilisse tasasesse ruumi dimensiooniga $N > 4$. Artiklis on analüüsitud sisestatud Einsteini ruumide infinitesimaalsete deformatsioonide geomeetrilisi omadusi. Sisestamine on defineeritud N-funktsiooniga $z^A(x^\mu)$, $A = 1, 2, \ldots, N$, $\mu = 0, 1, 2, 3$. Nende funktsioonide infinitesimaalseid deformatsioone saab arendada ritta väikese parameetri ε järgi: $z^A \to \tilde{z}^A = z^A + \varepsilon v^A + \varepsilon^2 w^A + \ldots$. Kõik geomeetrilised suurused on avaldatavad sisestamisfunktsioonide z^A ja nende deformatsioonide v^A, w^A jne kaudu. On lisatud nõue, et deformeeritud aegruum peab rahuldama Einsteini võrrandeid. Sel viisil saab konstrueerida Einsteini võrrandite ligikaudseid lahendeid. Lähemalt on vaadeldud gravitatsioonilisi tasalaineid kirjeldava lahendi tuletamist madalaimas mittetriviaalses lähenduses.