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MATHEMATICS

Pure spinor superfields, with application to $D = 3$ conformal models

Martin Cederwall

Fundamental Physics, Chalmers University of Technology, SE 412 96 Göteborg, Sweden; martin.cederwall@chalmers.se

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Abstract. I review and discuss the construction of supersymmetry multiplets and manifestly supersymmetric Batalin–Vilkovisky actions by using pure spinors, with emphasis on models with maximal supersymmetry. The special cases of $D = 3$, $N = 8$ (Bagger–Lambert–Gustavsson) and $N = 6$ (Aharony–Bergman–Jafferis–Maldacena) conformal models are treated in detail.

Key words: extended supersymmetry, superfields, pure spinors, conformal symmetry.

There is a close relationship between supermultiplets and pure spinors. The algebra of covariant fermionic derivatives in flat superspace is generically of the form

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c = -2\gamma^c_{\alpha\beta} D_c \,. \tag{1}$$

If a bosonic spinor λ^α is *pure*, i.e., if the vector part $(\lambda\gamma^a\lambda)$ of the spinor bilinear vanishes, the operator $Q = \lambda^\alpha D_\alpha$ becomes nilpotent and may be used as a Becchi–Rouet–Stora–Tyutin (BRST) operator. This is, schematically, the starting point for pure spinor superfields. (The details of the course depend on the actual space-time and the amount of supersymmetry. The pure spinor constraint may need to be further specified. Equation (1) may also contain more terms, due to super-torsion and curvature.) The cohomology of Q will consist of supermultiplets, which in case of maximal supersymmetry are on-shell. The idea of manifesting maximal supersymmetry off-shell by using pure spinor superfields $\Psi(x,\theta,\lambda)$ is to find an action whose equation of motion is $Q\Psi = 0$.

The fact that pure spinors had a role to play in maximally supersymmetric models was recognized early by Nilsson [1] and Howe [2,3]. Pure spinor superfields were developed with the purpose of covariant quantization of superstrings by Berkovits [4–7] and the cohomological structure was independently discovered in supersymmetric field theory and supergravity, originally in the context of higher-derivative deformations [8–17]. The present lecture only deals with pure spinors for maximally supersymmetric field theory.

The canonical example of pure spinors is in $D = 10$. There is only one non-gravitational supermultiplet, namely super-Yang–Mills, so this is what we expect to obtain. Expanding a field $\Psi(x,\theta,\lambda)$ in powers of λ, one has

$$\Psi(x,\theta,\lambda) = \sum_{n=0}^{\infty} \lambda^{\alpha_1} \ldots \lambda^{\alpha_n} \psi_{\alpha_1 \ldots \alpha_n}(x,\theta) \,. \tag{2}$$

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The implementation of the pure spinor constraint is as an Abelian gauge symmetry, where the generators $(\lambda\gamma^a\lambda)$ act multiplicatively. The field Ψ is defined modulo the ideal generated by the constraint. A 'canonical' representative of the gauge orbits is provided by superfields $\psi_{\alpha_1\ldots\alpha_n}(x,\theta)$ which, in addition to being symmetric, are completely γ-traceless, i.e., in the modules $(000n0)$ of the Lorentz algebra (where λ^α is in (00001) and D_α in (00010), the two spinor chiralities).

In order to calculate the cohomology, we start by finding the cohomology of zero-modes, x-independent fields. This cohomology is easy to calculate (a purely algebraic calculation), and gives information about the full cohomology. It is worth noting that the zero-mode cohomology (which clearly would have been empty for an unconstrained λ) may be read off from the partition function for a pure spinor. It is in one-to-one correspondence (for a concrete explanation of this fact, using the reducibility of the pure spinor constraint, see the appendix of [5] and [18]) with the six terms in the nominator of the partition function

$$Z(t)=\frac{1-10t^2+16t^3-16t^5+10t^6-t^8}{(1-t)^{16}}=\frac{(1+t^2)(1+4t+t^2)}{(1-t)^{11}}\,. \tag{3}$$

(This partition function only counts the dimension of the space of monomials in λ with degree of homogeneity p as the coefficient of t^p. A more refined partition function, specifying the actual Lorentz modules appearing, can of course be written down; for this I refer to [18].) The zero-mode cohomology is illustrated in Table 1. There, each column represents a field in the expansion 2, and the vertical direction is the expansion in θ. The columns have been shifted so that the components on the same row have the same dimension, i.e., so that Q acts horizontally. Since λ carries ghost number 1 and dimension $-1/2$, the component field $\psi_{\alpha_1\ldots\alpha_n}$ has ghost number $\mathrm{gh}(\Psi)-n$ and dimension $\dim(\Psi)+\frac{n}{2}$. It is natural to let $\mathrm{gh}(\Psi)=1$ and $\dim(\Psi)=0$ and take Ψ to be fermionic. Then the scalar (ghost number 1, dimension 0) in the first column is interpreted as the Yang–Mills ghost and the vector and spinor in the second column are interpreted as the fields of the super-Yang–Mills multiplet (the field ψ_α of ghost number 0 and dimension $1/2$ is the lowest-dimensional connection component A_α in a superfield treatment of super-Yang–Mills). The remaining fields are the corresponding antifields, in the Batalin–Vilkovisky (BV) sense. It is striking that one inevitably is led to the BV formalism. It of course exists also in a component formalism, but when one uses pure spinors, it is not optional. This means that any action formed in this formalism will be a BV action, and that the appropriate consistency relation (encoding the generalized gauge symmetry) is the master equation.

To go from the zero-mode cohomology to the complete cohomology, one easily convinces oneself that component fields in the modules contained in the zero-mode cohomology will be subject to differential

Table 1. The cohomology of the $D=10$ super-Yang–Mills complex

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
dim = 0	(00000)				
$\frac{1}{2}$	•	•			
1	•	(10000)	•		
$\frac{3}{2}$	•	(00001)	•	•	
2	•	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•	•
3	•	•	(10000)	•	•
$\frac{7}{2}$	•	•	•	•	•
4	•	•	•	(00000)	•
$\frac{9}{2}$	•	•	•	•	•

constraints in the modules of the zero-mode cohomology in the next column to the right. This gives the proper relations for the linearized on-shell super-Yang–Mills multiplet. (If a multiplet is an off-shell representation of supersymmetry, as is generically the case for half-maximal or lower supersymmetry, there will consequently be no anti-fields in the cohomology. These, instead, come in a separate pure spinor superfield [13].)

This far, we have not considered the actual solutions of pure spinor constraint, but rather regarded the pure spinor as a book-keeping device. When one wants to write down an action, this is no longer possible. For an action, a measure is needed. The linearized action should be '$\int \Psi Q \Psi$' for some suitable definition of '$\int$'. Clearly, '$\int$' must have ghost number -3. In the cohomology, there is a singlet at $\lambda^3\theta^5$. Defining a measure as a 'residue', picking the corresponding component, has the right ghost number, and also the correct dimension. However, it is singular, so components of Ψ with high enough power in λ or θ drop out of the putative action defined in this manner, and the equation of motion $Q\Psi = 0$ does not follow. Still, the corresponding tensorial structure can be used for an invariant integral over λ. It is clear from the partition function (3) that λ contains 11 degrees of freedom (out of the 16 for an unconstrained spinor). Explicit solution of the pure spinor constraint also shows that when imposed on a *complex* spinor, only five out of the ten constraints are independent (see, e.g., [5] for details). Defining the scalar at $\lambda^3\theta^5$ as $T_{\alpha_1\alpha_2\alpha_3}^{\beta_1\ldots\beta_{11}}\varepsilon_{\beta_1\ldots\beta_{16}}\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\theta^{\beta_{12}}\theta^{\beta_{13}}\theta^{\beta_{14}}\theta^{\beta_{15}}\theta^{\beta_{16}}$, where T thus is a Lorentz invariant tensor, one defines the conjugate invariant tensor $\tilde{T}^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\ldots\beta_{11}}$, and the integration is

$$[d\lambda]\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3} \sim \tilde{T}^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\ldots\beta_{11}} d\lambda^{\alpha_1}\wedge\ldots\wedge d\lambda^{\alpha_{11}}\ . \tag{4}$$

In [7], Berkovits solved the problem of how to make sense of this integration and use it as part of a non-singular measure for the pure spinor superspace. The solution involves a non-minimal set of pure spinor variables, which in addition to λ^α contains a bosonic conjugate spinor $\bar{\lambda}_\alpha$ (which in Euclidean signature can be viewed as the complex conjugate of λ^α) obeying $(\bar{\lambda}\gamma^a\bar{\lambda}) = 0$ and a fermionic spinor r_α with $(\bar{\lambda}\gamma^a r) = 0$. The new BRST operator is $Q = \lambda^\alpha D_\alpha + \frac{\partial}{\partial\bar{\lambda}_\alpha} r_\alpha$, and its cohomology is independent of $\bar{\lambda}$ and r. One assigns ghost number -1 and dimension $1/2$ to $\bar{\lambda}$ and ghost number 0 and dimension $1/2$ to r. The measure for $\bar{\lambda}$ is the complex conjugate to the one defined in Eq. (4) for λ, and for r:

$$[dr] \sim \star\tilde{T}^{\alpha_1\alpha_2\alpha_3}{}_{\beta_1\ldots\beta_{11}}\bar{\lambda}_{\alpha_1}\bar{\lambda}_{\alpha_2}\bar{\lambda}_{\alpha_3}\frac{\partial}{\partial r_{\beta_1}}\cdots\frac{\partial}{\partial r_{\beta_{11}}}\ . \tag{5}$$

Using these integration measures, and the ordinary ones for x and θ, we list the dimensions and ghost numbers for the theory after dimensional reduction to D dimensions in Table 2. So, the ghost numbers match, and also the dimensions ($\frac{1}{g^2}$ has dimension $D-4$ in D dimensions).

The λ and $\bar{\lambda}$ integrations are non-compact and need regularization. In [7] this is achieved, following [19], by the insertion of a factor $N = e^{\{Q,\chi\}}$. Since this differs from 1 by a Q-exact term, the regularization is independent of the choice of the fermion χ. The choice $\chi = -\bar{\lambda}_\alpha\theta^\alpha$ gives $N = e^{-\lambda^\alpha\bar{\lambda}_\alpha - r_\alpha\theta^\alpha}$ and regularizes the bosonic integrations at infinity. At the same time, it explains how the term at θ^5 is picked out; this follows after integration over r. N has definite ghost number 0 for the assignments for ghost number and dimension above (although any other assignment gives the correct ghost number and dimension for the non-Q-exact part).

An action for ten-dimensional super-Yang–Mills (or any dimensional reduction) can now be written in the Chern–Simons-like form [5]

$$S = \frac{1}{2g^2}\int <\Psi, Q\Psi + \frac{1}{3}[\Psi,\Psi]>_{\mathbf{adj}}. \tag{6}$$

Table 2. The dimensions and ghost numbers of the $D = 10$ measure

	gh#	dim
$d^D x$	0	$-D$
$d^{16}\theta$	0	8
$[d\lambda]$	8	-4
$[d\bar{\lambda}]$	-8	4
$[dr]$	-3	-4
Total	-3	$-(D-4)$

Note that there is no 4-point coupling. The component field 4-point coupling arises after elimination of unphysical components. One must, however, remember that this is a classical BV action. It obeys the classical master equation $(S,S) = 0$, where the anti-bracket takes the simple form

$$(A,B) = \int A < \frac{\overleftarrow{\delta}}{\delta\Psi(Z)} [dZ] \frac{\overrightarrow{\delta}}{\delta\Psi(Z)} >_{\mathbf{adj}} B \,. \tag{7}$$

In order to perform quantum calculations with path integral over Ψ, gauge fixing has to be implemented. This involves traditional gauge fixing (of the component gauge field), as well as elimination of the anti-fields. I will comment briefly on gauge fixing towards the end of the paper.

As already mentioned, pure spinor formulations are relevant for BV action formulations of any maximally supersymmetric model (exceptions being models containing self-dual tensors). I would now like to illustrate how they can be used for 3-dimensional conformal models. The pure spinor actions turn out to have a much simpler structure than the component actions. There has recently been much interest in conformal 3-dimensional theories. Following the discovery of the existence of a maximally supersymmetric ($N = 8$) interacting theory of scalar multiplets coupled to Chern–Simons, the Bagger–Lambert–Gustavsson (BLG) theory [20–23], much effort has been spent on trying to generalize the construction and to interpret it in terms of an anti-de Sitter (AdS) boundary model of multiple M2-branes. The interesting, but restrictive, algebraic structure of the model, containing a 3-algebra with antisymmetric structure constants, turned out to have only one finite-dimensional realization [24,25], possible to interpret in terms of two M2-branes [26,27] (see, however, [28–30] dealing with the infinite-dimensional solution related to volume-preserving diffeomorphisms in three dimensions).

It then became an urgent question how the stringent requirements in the BLG theory could be relaxed. There are different possibilities. One may let the scalar product on the matter representation be degenerate [31]. This works at the level of equations of motion, but does not allow for an action principle. One may also go one step further, and add further null directions to that degenerate case, which leads to scalar products with indefinite signature [32–34] (and consequently to matter kinetic terms with different signs). Or, finally, one may reduce the number of supersymmetries, specifically to $N = 6$, as proposed by Aharony, Bergman, Jafferis, and Maldacena (ABJM) [35] or maybe even to lower N [36,37]. The $N = 6$ models were further studied in [38–44] (among other papers). The literature on the subject is huge, and I apologize for omissions of references to relevant papers.

The superfield formulation of the BLG model was given in [45] (see also [46], where the on-shell superfields were constructed for the example of the BLG model based on the infinite-dimensional algebra of volume-preserving diffeomorphisms in three dimensions). A superfield formulation with $N = 1$ superfields was given in [47] and with $N = 2$ superfields in [48]. In [45] we constructed an action in an $N = 8$ pure spinor superspace formulation of the BLG model, which covers all situations with $N = 8$ above, except the ones with degenerate scalar product. The construction was essentially performed by using minimal pure spinor variables, and the issue of the integration measure was more or less neglected (the measure was assumed to exist). In the subsequent paper [49] also the $N = 6$ ABJM models were treated, and integration measures were defined by using non-minimal variables for both types of models.

Let us first briefly review the results of [45]. Since the BLG model is maximally supersymmetric, component formulations and also usual superspace formulations are on-shell. There is no finite set of auxiliary fields. A pure spinor treatment is necessary in order to write an action in a generalized BRST setting.

The Lorentz algebra in $D = 3$ is $so(1,2) \approx sl(2,\mathbb{R})$. The $N = 8$ theory has an $so(8)$ R-symmetry, and we choose the fermionic coordinates and derivatives to transform as $(\mathbf{2},\mathbf{8}_s) = (1)(0010)$ under $sl(2) \oplus so(8)$. This representation is real and self-conjugate. The pure spinors transform in the same representation and are written as $\lambda^{A\alpha}$, where A is the $sl(2)$ index and α the $so(8)$ spinor index. As usual, a BRST operator is formed as $Q = \lambda^{A\alpha} D_{A\alpha}$, D being the fermionic covariant derivative. The nilpotency of Q demands that

$$(\lambda^A \lambda^B) = 0, \tag{8}$$

where $(\ldots)$ denotes contraction of $so(8)$ spinor indices, since the superspace torsion has to be projected out. This turns out to be the full constraint[1]. As will soon be clear, it is essential that not only $(\lambda^A \sigma_{IJKL} \lambda^B)$ but also $\varepsilon_{AB}(\lambda^A \sigma_{IJ} \lambda^B)$ is left non-zero. These pure spinors are similar to those encountered in [50].

The 'pure spinor wave function' for the Chern–Simons field is a fermionic scalar Ψ of (mass) dimension 0 and ghost number 1. For the matter multiplet we have a bosonic field Φ^I in the $so(8)$ vector representation $(0)(1000)$ of dimension 1/2 and ghost number 0. In addition to the pure spinor constraint, the matter field is identified modulo transformations

$$\Phi^I \to \Phi^I + (\lambda^A \sigma^I \rho_A) \tag{9}$$

for arbitrary ρ. (This type of additional gauge invariance is typical of fields in some non-trivial module of the structure group. Without it the cohomology would be the tensor product of the module with the cohomology of a field in the trivial module.)

In this minimal pure spinor formulation the fields are expanded in power series in λ, i.e., in decreasing ghost number. The pure spinor partition function is easily calculated to be

$$Z_1(t) = \frac{1 - 3t^2 + 3t^4 - t^6}{(1-t)^{16}} = \frac{(1+t)^3}{(1-t)^{13}} \, . \tag{10}$$

The partition for a matter field is

$$Z_8 = \frac{8 - 16t + 16t^3 - 8t^4}{(1-t)^{16}} = 8\frac{(1+t)}{(1-t)^{13}} \, . \tag{11}$$

These expressions seem to imply that the number of independent degrees of freedom of a pure spinor is 13, i.e., that the pure spinor constraint in this case is irreducible. This is verified by a concrete solution of the constraint (for a complex λ) [49]. As for the $D = 10$ pure spinors, the partition functions can of course be refined to include not only the number of fields, but also modules of the structure group.

The field content (ghosts, fields and their antifields) is read off from the zero-mode BRST cohomology given in Tables 3 and 4 for the Chern–Simons and matter sectors, respectively.

We observe that the field content is the right one. In Ψ we find the ghost, the gauge connection, its antifield, and the antighost. The antifield has dimension 2 (as opposed to, e.g., $D = 10$ super-Yang–Mills, where it has dimension 3), indicating equations of motion that are first order in derivatives. It is quite striking that the (bosonic) Chern–Simons model has a natural supersymmetric off-shell extension, although the supersymmetry becomes trivial on-shell. It is not meaningful to talk about a gaugino field. In Φ we find the eight scalars ϕ^I, the fermions $\chi^{A\dot{\alpha}}$, and their antifields. In addition, the field Ψ transforms in the

[1] The vanishing of the 'torsion representation' – the vector part of the spinor bilinear – is necessary, but does not always give the full pure spinor constraint. One example where further constraints are needed is $N = 4$, $D = 4$ super-Yang–Mills theory.

Table 3. The cohomology of the scalar complex

gh# =	1	0	−1	−2	−3
dim = 0	(0)(0000)				
$\frac{1}{2}$	•	•			
1	•	(2)(0000)	•		
$\frac{3}{2}$	•	•	•	•	
2	•	•	(2)(0000)	•	•
$\frac{5}{2}$	•	•	•	•	•
3	•	•	•	(0)(0000)	•
$\frac{7}{2}$	•	•	•	•	•

Table 4. The cohomology of the vector complex

gh# =	0	−1	−2	−3	−4
dim = $\frac{1}{2}$	(0)(1000)				
1	(1)(0001)	•			
$\frac{3}{2}$	•	•	•		
2	•	(1)(0001)	•	•	
$\frac{5}{2}$	•	(0)(1000)	•	•	•
3	•	•	•	•	•
$\frac{7}{2}$	•	•	•	•	•

adjoint representation **adj** of some gauge group and Φ^I in some representation **R** of the gauge group. The corresponding indices are suppressed.

In order to derive the equations of motion for the physical component fields, one starts from the ghost number 0 part of the fields (i.e., $\Phi^I \to \phi^I(x,\theta)$ and $\Psi \to \lambda^\alpha A_\alpha(x,\theta)$, respectively) and examines the content of the θ expansion by repeated application of fermionic covariant derivatives, using the pure spinor constraint and the reducibility 9 when they occur. As a guideline one has the cohomology at ghost number 1; these representations are the only ones where an equation of motion may sit, for obvious reasons. In this manner, one derives the linearized component equations $\Box\phi^I = 0$, $\partial\!\!\!/\chi^A = 0$ for the scalar multiplet, and $dA = 0$ for the Chern–Simons field, and also the interacting equations from the actions below.

In [45] it was assumed that a non-degenerate measure can be formed by using a non-minimal extension of the pure spinor variables along the lines of [7]. This measure, including the 3-dimensional integration, should carry dimension 0 and ghost number -3, and should allow 'partial integration' of the BRST charge Q. It was then shown that the Lagrangian of the interacting model is of a very simple form, containing essentially a Chern–Simons-like term for the Chern–Simons field, minimally coupled to the matter sector:

$$S = \int <\Psi, Q\Psi + \frac{1}{3}[\Psi,\Psi]>_{\mathbf{adj}} + \int \frac{1}{2} M_{IJ} <\Phi^I, Q\Phi^J + \Psi\cdot\Phi^J>_{\mathbf{R}} \, . \tag{12}$$

The brackets denote (non-degenerate) scalar products on **adj** and **R**, $[\cdot,\cdot]$ the Lie bracket of the gauge algebra, and $T\cdot x$ the action of the Lie algebra element in the representation **R**. M_{IJ} is the pure spinor bilinear $\varepsilon_{AB}(\lambda^A\sigma_{IJ}\lambda^B)$, which is needed for several reasons: in order to contract the indices on the Φ's antisymmetrically, to get a Lagrangian of ghost number 3, and to ensure invariance in the equivalence classes defined by Eq. (9).

The invariances of the interacting theory (equivalent to the classical master equation $(S,S)=0$), generalizing the BRST invariance in the linearized case, are

$$\begin{aligned}\delta\Psi &= Q\Psi - [\Lambda,\Psi] - M_{IJ}\{\Phi^I,\Xi^J\}\,,\\ \delta\Phi^I &= -\Lambda\cdot\Phi^I + (Q+\Psi\cdot)\Xi^I\,,\end{aligned} \tag{13}$$

where Λ is an adjoint boson of dimension 0 and ghost number 0, and Ξ^I a fermionic vector in $\mathbf{R}$ of dimension 1/2 and ghost number -1. Here we also introduced the bracket $\{\cdot,\cdot\}$ for the formation of an adjoint from the antisymmetric product of two elements in $\mathbf{R}$, defined via $<x,T\cdot y>_{\mathbf{R}} = <T,\{x,y\}>_{\mathbf{adj}}$. The invariance with parameter Λ is manifest. The transformation with Ξ has to be checked. One then finds that the transformation of the matter field Φ gives a 'field strength' contribution from the anticommutator of the two factors $Q+\Psi$, which is cancelled against the variation of the Chern–Simons term. The single remaining term comes from the transformation of the Ψ in the covariant matter kinetic term, and it is proportional to $M_{IJ}M_{KL}<\{\Phi^I,\Phi^J\},\{\Phi^K,\Xi^L\}>_{\mathbf{adj}}$. Due to the pure spinor constraint, $M_{[IJ}M_{KL]}=0$. This was shown in [45], by using the simple observation that the only $sl(2)$ singlet at the fourth power of λ is in the $sl(2)\oplus so(8)$ representation $(0)(0200)$ – the four-index antisymmetric tensors $(0)(0020)$ or $(0)(0002)$ do not occur. So if the structure constants of the 3-algebra defined by $<\{x,a\},\{b,c\}>_{\mathbf{adj}} = <x,[\![a,b,c]\!]>_{\mathbf{R}}$ are antisymmetric, this term vanishes. It was also checked that the commutator of two Ξ-transformations gives a Λ-transformation together with a transformation of the type (9). In this way, one is naturally led to the 3-algebra structure with a minimal amount of input, essentially a 'minimal coupling'. I would like to stress that although the pure spinor action contains at most 3-point couplings, the full component action with up to 6-point interactions will arise when unphysical component fields are eliminated.

For the $N=6$ ABJM models, the results are very similar. Due to lack of space, I will not go into details, but refer to [49]. The end result is a weaker condition on the structure constants of the '3-algebra', which is just the appropriate one [42]. The classification of such algebraic structures was performed in [43]. It is satisfactory that the structure of the pure spinors in both cases gives the necessary and sufficient algebraic structure by the vanishing of a single term in the transformation of minimally coupled matter.

In both the $N=8$ and $N=6$ theories in $D=3$, the naïve measure sits at $\lambda^3\theta^3$. In analogy with the 10-dimensional case, we need the number of irreducible constraints on the pure spinors to equal the number of θ's. Indeed, as mentioned, the constraints, which in both cases sit in the vector representation of $so(1,2)$, turn out to be irreducible, which is straightforward to check (explicit solutions were given in [49]).

We can write the invariant tensors as

$$\varepsilon_{abc}(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta) = T_{(A_1\alpha_1,A_2\alpha_2,A_3\alpha_3)[B_1\beta_1,B_2\beta_2,B_3\beta_3]}\lambda^{A_1\alpha_1}\lambda^{A_2\alpha_2}\lambda^{A_3\alpha_3}\theta^{B_1\beta_1}\theta^{B_2\beta_2}\theta^{B_3\beta_3} \tag{14}$$

in the $N=8$ case, and as a similar expression when $N=6$. The integration measures are constructed by using these invariant tensors in a manner completely analogous to the measure in $D=10$.

Let us examine the dimensions and ghost numbers of the total measures. The analogies of Table 2 are obtained by simple counting (the $N=6$ case is included for completeness) and are given in Table 5.

In both cases we get a non-degenerate measure of dimension 0 and ghost number -3, as desired for a conformal theory. Also here, the measures of course have to be regularized in the same way as in [7]. We insert a factor $N=e^{\{\mathscr{Q},\chi\}}$, where $\chi=-\mu_{A\alpha}\theta^{A\alpha}$ for $N=8$ (and similarly for $N=6$).

To conclude, I have presented manifestly supersymmetric formulations of the $N=8$ BLG models and the $N=6$ ABJM models. I have also performed a detailed analysis of the pure spinor constraints and provided proper actions based on non-degenerate measures on non-minimal pure spinor spaces. I hope that these formulations may be helpful in the future, e.g., for the investigation of quantum properties [51,52] of the models. In order to perform path integrals, one has to gauge fix. Gauge fixing in the pure spinor formalism for the superparticle includes the 'b-ghost', with the property $\{Q,b\}=\square$. The b-ghost is a composite

Table 5. The dimensions and ghost numbers of the $N = 8$ and $N = 6$ measures

	$N=8$		$N=6$	
	gh#	dim	gh#	dim
d^3x	0	−3	0	−3
$[d\theta]$	0	8	0	6
$[d\lambda]$	10	−5	6	−3
$[d\bar{\lambda}]$	−10	5	−6	3
$[dr]$	−3	−5	−3	−3
Total	−3	0	−3	0

operator in the pure spinor formalism, since $p^2 = 0$ is not an independent constraint. This operator has singularities at $\lambda = 0$, which need to be regularized. Proposals for resolving this issue and allowing for calculation of string amplitudes at arbitrary loop level have been made in [53,54], but lead to complicated expressions. It is possible that some simpler approach exists.

I believe that much more is to be learnt from pure spinor superspace formulations, especially of maximally supersymmetric theories. One very interesting example, largely unexplored, is the issue of such formulations of supergravities. I think that the treatment of the scalar multiplet actions in the present work may contain clues to supergravity, both considering the extra gauge invariances and the extra factors of λ in the action. Work is in progress.

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Puhtad spiinor-superväljad ja nende rakendused $D = 3$ konformsetes mudelites

Martin Cederwall

Vaatluse alla on võetud supersümmeetriliste multiplettide ja Batalini-Vilkovõsski mõjufunktsionaali konstrueerimine puhaste spiinorite abil, pöörates erilist tähelepanu maksimaalse supersümmeetriaga mudelitele. Detailselt on käsitletud $D = 3$, $N = 8$ (Baggeri-Lamberti-Gustavssoni) ja $N = 6$ (Aharony-Bergmani-Jafferise-Maldacena) konformseid mudeleid. Suur osa esitatust on leitav ka e-printides arXiv:0808.3242 ja arXiv:0809.0318. Artikkel on trükiversioon autori ettekandest IV Baltimaade-Põhjamaade seminaril "Algebra, geomeetria ja matemaatiline füüsika" (Tartu, 9.–11. oktoober 2008).