Proceedings of the Estonian Academy of Sciences,
2010, **59**, 4, 256–264
doi: 10.3176/proc.2010.4.02
Available online at www.eap.ee/proceedings

MATHEMATICS

# Generalization of connection based on the concept of graded $q$-differential algebra

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Received 29 September 2009, accepted 9 October 2009

**Abstract.** We propose a generalization of the concept of connection form by means of a graded $q$-differential algebra $\Omega_q$, where $q$ is a primitive $N$th root of unity, and develop the concept of curvature $N$-form for this generalization of the connection form. The Bianchi identity for a curvature $N$-form is proved. We study an $\Omega_q$-connection on module and prove that every projective module admits an $\Omega_q$-connection. If the module is equipped with a Hermitian structure, we introduce a notion of an $\Omega_q$-connection consistent with the Hermitian structure.

**Key words:** $q$-connection on module, connection form, covariant $N$-differential, Hermitian connection, Hermitian module, generalized cohomologies, $N$-complex.

## 1. INTRODUCTION

It is well known that the concepts of connection and its curvature are basic elements of the theory of fibre bundles and play an important role not only in modern differential geometry, but also in modern theoretical physics, namely in the gauge field theory. The development of a theory of connections has been closely related to the development of theoretical physics. The advent of supersymmetric field theories in the 1970s gave rise to interest towards $\mathbb{Z}_2$-graded structures which became known in theoretical physics under the name of superstructures. This direction of development has led to the concept of superconnection which appeared in [12]. The emergence of noncommutative geometry in the 1980s was a powerful spur to the development of the theory of connections on modules [5,6,9,11]. A basic concept used in the theory of connections on modules is the notion of graded differential algebra. This notion has been generalized to the notion of graded $q$-differential algebra, where $q$ is a primitive $N$th root of unity (see papers [7,8,10]).

In Section 2 and Section 3 we give a short overview of $N$-structures, such as $N$-differential module, cochain $N$-complex, generalized cohomologies of an $N$-complex, and graded $q$-differential algebra. In Section 4 we introduce the notion of connection form in a graded $q$-differential algebra and covariant $N$-differential, which can be viewed as analogues of the connection form in a graded differential algebra described in [13]. In order to study the structure of a connection form in a graded $q$-differential algebra, we introduce an algebra of polynomials in the variables $\mathfrak{d}, a_1, a_2, \ldots$ and prove the power expansion formula for an $n$th power of the operator $\hat{\mathfrak{d}}_a = \hat{\mathfrak{d}} + a_1$. Applying this formula, we show that the $N$th power of the covariant $N$-differential is the operator of multiplication by an element $F_A^{(N)}$, which we then define as the curvature $N$-form of a connection form $A$. We also study the concept of $\Omega_q$-connection on module, where

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$\Omega_q$ is a graded $q$-differential algebra, introduced in [2–4], and define the notions, such as dual $\Omega_q$-connection and $\Omega_q$-connection consistent with the Hermitian structure of a module.

## 2. $N$-COMPLEX

Let $K$ be a commutative ring with a unit and $E$ be a left $K$-module. The module $E$, endowed with an endomorphism $d$ satisfying $d^2 = 0$, is referred to as a differential module and an endomorphism $d$ as its differential. If $K$ is a field, then the differential module $E$ will be referred to as a differential vector space. From the property of the differential $d^2 = 0$ it follows that $\mathrm{Im}\, d \subset \mathrm{Ker}\, d$ and one can measure the non-exactness of the sequence $E \xrightarrow{d} E \xrightarrow{d} E$ by the quotient module $H(E) = \mathrm{Ker}\, d/\mathrm{Im}\, d$, which is referred to as the homology of the differential module $E$. Let $E, F$ be differential modules, respectively with differentials $d, d'$. A homomorphism of differential modules is a homomorphism of modules $\phi : E \longrightarrow F$ satisfying $\phi \circ d = d' \circ \phi$. Obviously $\phi\,(\mathrm{Im}\, d) \subset \mathrm{Im}\, d'$, $\phi\,(\mathrm{Ker}\, d) \subset \mathrm{Ker}\, d'$, and $\phi$ induces the homomorphism $\phi_* : H(E) \longrightarrow H(F)$ in homology. Given an exact sequence of differential modules

$$0 \rightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \rightarrow 0,$$

one can construct a homomorphism $\partial : H(G) \rightarrow H(E)$ such that the triangle

$$\begin{tikzcd} & H(F) \arrow[dr, "\psi_*"] & \\ H(E) \arrow[ur, "\varphi_*"] & & H(G) \arrow[ll, "\partial"] \end{tikzcd} \tag{1}$$

is exact [8].

A cochain complex is a $\mathbb{Z}$-graded differential module $E = \oplus_{i \in \mathbb{Z}} E^i$ whose differential $d$ has degree 1, which means $d : E^n \rightarrow E^{n+1}$. The homology $H(E)$ of a cochain complex inherits a $\mathbb{Z}$-graded structure of the cochain complex $E$. Hence $H(E) = \oplus_{i \in \mathbb{Z}} H^i(E)$, where $H^i(E) = \mathrm{Ker}\, d \cap E^i/\mathrm{Im}\, d \cap E^i$, and $H(E)$ is usually referred to as a cohomology of the cochain complex $E$. Given an exact sequence of cochain complexes

$$0 \rightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \rightarrow 0,$$

one can construct by means of (1) the following exact sequence:

$$\ldots \xrightarrow{\partial} H^n(E) \xrightarrow{\phi_*} H^n(F) \xrightarrow{\psi_*} H^n(G) \xrightarrow{\partial} H^{n+1}(E) \rightarrow \ldots$$

Let $N \geq 2$ be a positive integer. The left $K$-module $E$ is said to be an $N$-differential module with $N$-differential $d$ if $d$ is an endomorphism of $E$ satisfying $d^N = 0$. Obviously, an $N$-differential module can be viewed as a generalization of the concept of differential module to any integer $N \geq 2$. If $K$ is a field, an $N$-differential module will be referred to as an $N$-differential vector space.

For each integer $m$ with $1 \leq m \leq N-1$ we can define the submodules $Z_m(E) = \mathrm{Ker}\,(d^m)$ and $B_m(E) = \mathrm{Im}\,(d^{N-m})$. It follows from the equation $d^N = 0$ that $B_m(E) \subset Z_m(E)$ and the quotient modules $H_m(E) := Z_m(E)/B_m(E)$ are called the (generalized) homology of the $N$-differential module $E$. As in the case of the homology of a differential module, one can prove a proposition analogous to (1), which asserts that if $0 \rightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \rightarrow 0$ is an exact sequence of $N$-differential modules, then there exist homomorphisms $\partial : H_m(G) \longrightarrow H_{N-m}(E)$ for $m \in \{1, 2, \ldots, N-1\}$ such that the following hexagons of homomorphisms

$$
\begin{tikzcd}
& H_{(n)}(F) \arrow[r, "\psi_*"] & H_{(n)}(G) \arrow[dr, "\partial"] & \\
H_{(n)}(E) \arrow[ur, "\varphi_*"] & & & H_{(N-n)}(E) \arrow[dl, "\varphi_*"] \\
& H_{(N-n)}(G) \arrow[ul, "\partial"] & H_{(N-n)}(F) \arrow[l, "\psi_*"] &
\end{tikzcd}
$$

are exact for $n \in \{1,2,\dots,N-1\}$ [8]. A cochain $N$-complex of modules or simply an $N$-complex is a $\mathbb{Z}$-graded $N$-differential module $E = \oplus_{k\in\mathbb{Z}} E^k$ with a homogeneous $N$-differential $d$ of degree 1. If $E$ is an $N$-complex, then its cohomologies $H_m(E)$ are $\mathbb{Z}$-graded modules, i.e. $H_m(E) = \oplus_{n\in\mathbb{Z}} H^n_m(E)$, where

$$H^n_m(E) = \mathrm{Ker}\,(d^m : E^n \longrightarrow E^{n+m})/d^{N-m}(E^{n+m-N}).$$

It should be noted that many notions related to $N$-complexes depend only on the underlying $\mathbb{Z}_N$-graduation. For this purpose we define a $\mathbb{Z}_N$-complex to be a $\mathbb{Z}_N$-graded $N$-differential module with $N$-differential $d$ of degree 1.

## 3. GRADED $q$-DIFFERENTIAL ALGEBRA

Let $\Omega = \oplus_{n\in\mathbb{Z}} \Omega^n$ be a unital associative graded $\mathbb{C}$-algebra. The subspace of elements of grading zero $\Omega^0 \subset \Omega$ is the subalgebra of $\Omega$, which we denote by $\mathfrak{A}$, i.e. $\mathfrak{A} = \Omega^0$. Any subspace $\Omega^k \subset \Omega$ of elements of grading $k \in \mathbb{Z}$ is the $\mathfrak{A}$-bimodule. A graded differential algebra is a unital associative graded $\mathbb{C}$-algebra equipped with a linear mapping $d$ of degree 1 such that the sequence

$$\dots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \dots$$

is a cochain complex and $d$ is an antiderivation, i.e. it satisfies the graded Leibniz rule

$$d(\omega\cdot\theta) = d\omega\cdot\theta + (-1)^k \omega\cdot d\theta,$$

where $\omega \in \Omega^k, \theta \in \Omega$. Let us mention that if $\Omega$ is a graded differential algebra, then $\mathrm{Ker}\,d$ is the graded unital subalgebra of $\Omega$, whereas $\mathrm{Im}\,d$ is the graded two-sided ideal of $\mathrm{Ker}\,d$, so the cohomology $H(\Omega)$ is the unital associative graded algebra. Obviously $\mathfrak{A} \xrightarrow{d} \Omega^1$ is a first-order differential calculus over the algebra $\mathfrak{A}$. In what follows we shall call $\Omega$ a differential calculus over the algebra $\mathfrak{A}$.

Making use of the notion of $N$-complex described in the previous section, one can generalize the concept of graded differential algebra [8,10]. Let $K$ be the field of complex numbers $\mathbb{C}$ and $q$ be a primitive $N$th root of unity, where $N \geq 2$. A graded $q$-differential algebra is a unital associative $\mathbb{Z}$-graded ($\mathbb{Z}_N$-graded) $\mathbb{C}$-algebra $\Omega_q = \oplus_k \Omega^k_q$ endowed with a linear mapping $d$ of degree one such that the sequence

$$\dots \xrightarrow{d} \Omega^{k-1}_q \xrightarrow{d} \Omega^k_q \xrightarrow{d} \Omega^{k+1}_q \xrightarrow{d} \dots$$

is an $N$-complex with $N$-differential $d$ satisfying the graded $q$-Leibniz rule

$$d(\omega\cdot\theta) = d\omega\cdot\theta + q^k \omega\cdot d\theta,$$

where $\omega \in \Omega^k_q, \theta \in \Omega_q$. As in the case of a graded differential algebra, the subspace of elements of grading zero $\mathfrak{A} = \Omega^0_q$ is the subalgebra of the graded $q$-differential algebra $\Omega_q$. By analogy with the terminology used in the case of a graded differential algebra, we shall call $\Omega_q$ a $q$-differential calculus over the algebra $\mathfrak{A}$. Let us mention that $d:\mathfrak{A}\longrightarrow\Omega^1_q$ is a first-order differential calculus over the algebra $\mathfrak{A}$.

Let us remind that a $q$-graded centre of an associative unital graded $\mathbb{C}$-algebra $\mathscr{A}=\oplus_{k\in\mathbb{Z}}\mathscr{A}^k$ is the graded subspace $Z(\mathscr{A})=\oplus_{k\in\mathbb{Z}}Z^k(\mathscr{A})$ of $\mathscr{A}$ generated by the homogeneous elements $v\in\mathscr{A}^k$, where $k\in\mathbb{Z}$, satisfying $vw=q^{km}wv$ for any $w\in\mathscr{A}^m$, $m\in\mathbb{Z}$. The graded $q$-centre $Z(\mathscr{A})$ is the graded subalgebra of $\mathscr{A}$. A graded $q$-derivation of degree $k\in\mathbb{Z}$ of $\mathscr{A}$ is a homogeneous linear mapping $D:\mathscr{A}^m\longrightarrow\mathscr{A}^{m+k}$ satisfying the graded $q$-Leibniz rule $D(vw)=D(v)\,w+q^{km}v\,D(w)$, where $v\in\mathscr{A}^m$. If $v\in\mathscr{A}^k$ is a homogeneous element, then $v$ determines the graded $q$-derivation of degree $k$ by means of a graded $q$-commutator as follows: $\mathrm{ad}_q(v)w=[v,w]_q=vw-q^{km}wv$, where $w\in\mathscr{A}^m$, and $\mathrm{ad}_q(v)$ is called an inner graded $q$-derivation of degree $k$ of $\mathscr{A}$.

It is proved in [1] that if $\mathscr{A}$ is an associative unital graded $\mathbb{C}$-algebra and $v$ is an element of grading one of this algebra satisfying $v^N\in Z(\mathscr{A})$, where $N\geq 2$, then the inner graded $q$-derivation $d_v=\mathrm{ad}_q(v):\mathscr{A}^k\longrightarrow\mathscr{A}^{k+1}$ is the $N$-differential of an algebra $\mathscr{A}$ and $\mathscr{A}$ is the graded $q$-differential algebra with respect to $d$. Making use of this theorem, we can endow a generalized Clifford algebra with a structure of graded $q$-differential algebra. Indeed, let us remind that a generalized Clifford algebra $\mathfrak{C}^N_p$ is an associative unital $\mathbb{C}$-algebra generated by $\gamma_1,\gamma_2,\ldots,\gamma_p$, which are subjected to the relations

$$\gamma_i\gamma_j=q^{\mathrm{sg}(j-i)}\gamma_j\gamma_i,\quad \gamma_i^N=\mathbb{1},\quad i,j=1,2,\ldots,p,$$

where $\mathbb{1}$ is the identity element of $\mathfrak{C}^N_p$. If we assign grading zero to the identity element $\mathbb{1}$ and grading one to each generator $\gamma_i$, then $\mathfrak{C}^N_p$ becomes the $\mathbb{Z}_N$-graded algebra. It is easy to verify that the $N$th power of any linear combination of generators $v=\sum_\mu\lambda_\mu\gamma_\mu,\lambda_\mu\in\mathbb{C}$ belongs to the graded $q$-centre of $\mathfrak{C}^N_p$, i.e. $v^N\in Z(\mathfrak{C}^N_p)$. Hence, $d_v=\mathrm{ad}_q(v)$ is the $N$-differential of $\mathfrak{C}^N_p$ and $\mathfrak{C}^N_p$ is the graded $q$-differential algebra.

## 4. GENERALIZATION OF CONNECTION

In this section we propose a generalization of the concept of connection form and connection on a module by means of the notion of graded $q$-differential algebra [2–4]. We begin with an algebra of polynomials on two variables, which we will use later to prove propositions describing the structure of the curvature of a connection form.

Let $\mathfrak{P}[\mathfrak{d},a]$ be the algebra of polynomials in $\mathfrak{d},a_1,a_2,\ldots$ with coefficients in $\mathbb{C}$ and the variables $\mathfrak{d},a_1,a_2,\ldots$ be subjected to the relations

$$\mathfrak{d}a_i=q\,a_i\mathfrak{d}+a_{i+1},\qquad i=1,2,\ldots,\tag{2}$$

where $q$ is a complex number. Let $\mathfrak{P}[\mathfrak{d}]$ be the subalgebra of $\mathfrak{P}[\mathfrak{d},a]$, generated by the variable $\mathfrak{d}$, and $\mathfrak{P}[a]$ be the subalgebra of $\mathfrak{P}[\mathfrak{d},a]$, freely generated by the variables $a_1,a_2,\ldots,a_n,\ldots$ . We equip the algebra of polynomials $\mathfrak{P}[\mathfrak{d},a]$ with the $\mathbb{N}$-graduation, by assigning grading one to the generator $\mathfrak{d}$ and grading $i$ to the generator $a_i$. It should be mentioned that this $\mathbb{N}$-graded structure of $\mathfrak{P}[\mathfrak{d},a]$ induces the $\mathbb{N}$-graded structures on the subalgebras $\mathfrak{P}[\mathfrak{d}],\mathfrak{P}[a]$. We will call $\mathfrak{P}[\mathfrak{d},a]$ the $N$-reduced algebra of polynomials and denote it by $\mathfrak{P}_N[\mathfrak{d},a]$ if $q$ is a primitive $N$th root of unity and $\mathfrak{d}$ obeys the additional relation $\mathfrak{d}^N=0$. It can be shown by means of (2) that if $q$ is a primitive $N$th root of unity and $\mathfrak{d}^N=0$, then $a_n=0$ for any integer $n\geq N$. Indeed, making use of the commutation relations (2), we can express any variable $a_n,n\geq 2$, in terms of $\mathfrak{d},a_1$ as the polynomial

$$a_n=\sum_{k+m=n}(-1)^m q^{\frac{(m+1)m}{2}}\left[\begin{array}{c}n\\m\end{array}\right]_q\mathfrak{d}^k a_1\mathfrak{d}^m.\tag{3}$$

If $n=N$ in (3), then the first and last terms in (3) vanish because of $\mathfrak{d}^N=0$; other terms are zero because of the vanishing of $q$-binomial coefficients, provided $q$ is a primitive $N$th root of unity. Hence, $a_N=0$, and the

$N$-reduced algebra of polynomials $\mathfrak{P}_N[\mathfrak{d},a]$ is an algebra over $\mathbb{C}$ generated by $\mathfrak{d}, a_1, a_2, \ldots, a_{N-1}$, which are subjected to the commutation relations

$$\begin{aligned}\mathfrak{d}a_1 &= q\,a_1\mathfrak{d} + a_2,\\ \mathfrak{d}a_2 &= q^2\,a_2\mathfrak{d} + a_3,\\ &\ldots\\ \mathfrak{d}a_{N-2} &= q^{N-1}\,a_{N-2}\mathfrak{d} + a_{N-1},\\ \mathfrak{d}a_{N-1} &= a_{N-1}\mathfrak{d},\end{aligned}$$

and $\mathfrak{d}^N = 0$. We denote by $\mathfrak{P}_N[a]$ the subalgebra of $\mathfrak{P}_N[\mathfrak{d},a]$ generated by $a_1, a_2, \ldots, a_{N-1}$.

Let us consider the algebra $\mathfrak{P}[\mathfrak{d},a]$ and its subalgebra $\mathfrak{P}[a]$ generated by $a_i, i \geq 1$. We assign a linear operator $\hat{\mathfrak{d}} : \mathfrak{P}[a] \longrightarrow \mathfrak{P}[a]$ to the generator $\mathfrak{d}$ by putting

$$\hat{\mathfrak{d}}(1) = 0,\ \hat{\mathfrak{d}}(a_i) = a_{i+1},\ \hat{\mathfrak{d}}(a_i a_j) = a_{i+1}a_j + q^i a_i a_{j+1}.$$

Evidently $\hat{\mathfrak{d}}$ is the graded $q$-differential on the subalgebra $\mathfrak{P}[a]$, i.e. $\hat{\mathfrak{d}}$ satisfies the graded $q$-Leibniz rule with respect to $\mathbb{N}$-graded structure of $\mathfrak{P}[a]$. It is easy to see that in the case of the $N$-reduced algebra of polynomials $\mathfrak{P}_N[\mathfrak{d},a]$, the $N$th power of $\hat{\mathfrak{d}} : \mathfrak{P}_N[a] \longrightarrow \mathfrak{P}_N[a]$ is zero, i.e. $\hat{\mathfrak{d}}^N = 0$. Hence, $\mathfrak{P}_N[a]$ is the graded $q$-differential algebra and $\hat{\mathfrak{d}}$ is its $N$-differential.

Now, for any integer $n \geq 1$ we define the polynomials $p^{(n)} \in \mathfrak{P}[\mathfrak{d},a]$, $f_a^{(n)} \in \mathfrak{P}[a]$ and the operator $\hat{\mathfrak{d}}_a : \mathfrak{P}[a] \longrightarrow \mathfrak{P}[a]$ of grading one by

$$p^{(n)} = (\mathfrak{d} + a_1)^n, \quad f_a^{(k+1)} = \hat{\mathfrak{d}}_a^k(a_1), \quad \hat{\mathfrak{d}}_a(p) = \hat{\mathfrak{d}}(p) + a_1 p, \quad \forall p \in \mathfrak{P}[a]. \tag{4}$$

For the first values of $k$ the straightforward computation of polynomials $f_a^{(k)}$ by means of the recurrent relation $f_a^{(k+1)} = \hat{\mathfrak{d}}_a(f_a^{(k)})$ gives

$$\begin{aligned}
f_a^{(2)} &= a_2 + a_1^2, &(5)\\
f_a^{(3)} &= a_3 + a_2\,a_1 + [2]_q\,a_1\,a_2 + a_1^3, &(6)\\
f_a^{(4)} &= a_4 + a_3\,a_1 + [3]_q\,a_1 a_3 + [3]_q\,a_2^2\\
&\quad + a_2 a_1^2 + [3]_q\,a_1^2 a_2 + [2]_q\,a_1 a_2 a_1 + a_1^4, &(7)\\
f_a^{(5)} &= a_5 + a_4 a_1 + [4]_q\,a_1 a_4 + [4]_q\,a_3 a_2\\
&\quad + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q a_2 a_3 + a_3 a_1^2 + [3]_q\,a_2^2 a_1 + [4]_q\,a_2 a_1 a_2\\
&\quad + [2]_q\,[4]_q\,a_1 a_2^2 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q a_1^2 a_3 + [3]_q\,a_1 a_3 a_1\\
&\quad + [2]_q\,a_1 a_2 a_1^2 + [3]_q\,a_1^2 a_2 a_1 + a_2 a_1^3 + [4]_q\,a_1^3 a_2 + a_1^5. &(8)
\end{aligned}$$

We assign the operator $\hat{p}^{(n)} : \mathfrak{P}[a] \longrightarrow \mathfrak{P}[a]$ to the polynomial $p^{(n)}$, replacing the variable $\mathfrak{d}$ in $p^{(n)}$ by the operator $\hat{\mathfrak{d}}$. Evidently, $\hat{p}^{(n)} = \hat{\mathfrak{d}}_a^n$ and $f_a^{(n+1)} = \hat{p}^{(n)}(a_1)$. Our aim now is to find a power expansion for polynomials $p^{(n)}$ with respect to variables $\mathfrak{d}, a_1, a_2, \ldots, a_n$. It is obvious that making use of the commutation relations (2), we can rearrange the factors in each summand of this expansion by removing all $\mathfrak{d}$'s to the right.

**Theorem 4.1.** *Each polynomial $p^{(n)}$ can be expanded with respect to variables of the algebra $\mathfrak{P}[\mathfrak{d},a]$ as follows*:

$$p^{(n)} = \sum_{k+m=n} \begin{bmatrix} n \\ k \end{bmatrix}_q f_a^{(m)}\,\mathfrak{d}^k = \mathfrak{d}^n + [n]_q f_a^{(1)}\mathfrak{d}^{n-1} + \ldots + [n]_q f_a^{(n-1)}\mathfrak{d} + f_a^{(n)},$$

*where* $f_a^{(n)} = \hat{p}^{(n-1)}(a_1)$. *In the case of N-reduced algebra of polynomials* $\mathfrak{P}_N[\mathfrak{d},a]$, *the operator* $\hat{p}^{(N)} = \hat{\mathfrak{d}}_a^N : \mathfrak{P}_N[a] \longrightarrow \mathfrak{P}_N[a]$, *induced by the polynomial* $p^{(N)}$, *is the operator of multiplication by* $f_a^{(N)}$.

Let $\Omega_q$ be a $\mathbb{Z}_N$-graded $q$-differential algebra with $N$-differential $d$, where $q$ is an $N$th primitive root of unity, and $\mathfrak{A} = \Omega_q^0$ be the subalgebra of elements of grading zero. We will call an element of grading one $A \in \Omega_q^1$ a connection form in a graded $q$-differential algebra $\Omega_q$. Since $d$ is an $N$-differential, which means that $d^n \neq 0$ for $1 \leq n \leq N-1$, if we successively apply it to a connection form $A$, we get the sequence of elements $A, dA, d^2A, \ldots, d^{N-1}A$, where $d^nA \in \Omega_q^{n+1}$. Let us denote by $\Omega_q[A]$ the graded subalgebra of $\Omega_q$ generated by $A, dA, d^2A, \ldots, d^{N-1}A$. The linear operator of degree one $d_A = d + A : \Omega^i \longrightarrow \Omega^{i+1}$ will be called a covariant $N$-differential induced by a connection form $A$. For any integer $n = 1, 2, \ldots, N$ we define the polynomial $F_A^{(n)} \in \Omega_q[A]$ by the formula $F_A^{(n)} = d_A^{n-1}(A)$.

Now we apply the $N$-reduced algebra of polynomials $\mathfrak{P}_N[\mathfrak{d},a]$, constructed above to study the structure of a $k$th power of the covariant $N$-differential $d_A$. Indeed, it is easy to see that we can identify an $N$-differential $d$ in $\Omega_q$ with the variable $\mathfrak{d}$ in $\mathfrak{P}_N[\mathfrak{d},a]$, a connection form $A$ in $\Omega_q$ with the variable $a_1$ in $\mathfrak{P}_N[\mathfrak{d},a]$ and $d^nA \in \Omega_q$ with $a_{n+1} \in \mathfrak{P}_N[\mathfrak{d},a]$. Then the commutation relations (4) between $\mathfrak{d}, a_i$ are equivalent to the graded $q$-Leibniz rule for an $N$-differential $d$, and $f_a^{(n)}$ can be identified with $F_A^{(n)}$. Consequently, from Theorem 4.1 we obtain

**Proposition 4.2.** *For any integer* $1 \leq n \leq N$ *the nth power of the covariant N-differential* $d_A$ *can be expanded as follows*:

$$(d_A)^n = \sum_{k+m=n} \begin{bmatrix} n \\ k \end{bmatrix}_q F_A^{(m)}\, d^k = d^n + [n]_q F_A^{(1)} d^{n-1} + \ldots + [n]_q F_A^{(n-1)} d + F_A^{(n)},$$

*where* $F_A^{(n)} = (d_A)^{n-1}(A)$. *Particularly, if* $n = N$, *then the Nth power of the covariant N-differential* $d_A$ *is the operator of multiplication by the element* $F_A^{(N)}$ *of degree zero.*

Proposition 4.2 allows us to define the curvature of a connection form $A$ as follows.

**Definition 4.3.** *The curvature N-form of a connection form A is the element of grading zero* $F_A^{(N)} \in \mathfrak{A}$.

It is easy to see that in the particular case of a graded differential algebra ($N = 2, q = -1$) with differential $d$ satisfying $d^2 = 0$ the above definition yields a connection form $A$ and its curvature $F_A^{(2)} = dA + A^2$ as elements of a graded differential algebra, respectively, of grading one and two [13].

**Proposition 4.4.** *For any connection form A in a graded q-differential algebra* $\Omega_q$ *the curvature N-form* $F_A^{(N)}$ *satisfies the Bianchi identity*

$$dF_A^{(N)} + [A, F_A^{(N)}]_q = 0.$$

*Proof.* Indeed, we have

$$\begin{aligned} dF_A^{(N)} + [A, F_A^{(N)}]_q &= dF_A^{(N)} + AF_A^{(N)} - q^N F_A^{(N)} A \\ &= (d_A) F_A^{(N)} - F_A^{(N)} A \\ &= (d_A)\Big((d_A)^{N-1} A\Big) - F_A^{(N)} A \\ &= (d_A)^N A - F_A^{(N)} A = F_A^{(N)} A - F_A^{(N)} A = 0. \end{aligned}$$

□

From (5)–(8) we obtain the expressions for curvature form

$$\begin{aligned} F_A^{(2)} &= dA + A^2, \\ F_A^{(3)} &= d^2A + dAA + [2]_q A\,dA + A^3, \\ F_A^{(4)} &= d^3A + (d^2A)A + [3]_q A\,(d^2A) + [3]_q\,(dA)^2 \\ &\quad + dAA^2 + [3]_q A^2\,dA + [2]_q A\,dAA + A^4, \\ F_A^{(5)} &= d^4A + (d^3A)A + [4]_q A\,(d^3A) + [4]_q\,(d^2A)\,dA \\ &\quad + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q dA\,(d^2A) + (d^2A)A^2 + [3]_q\,(dA)^2A \\ &\quad + [4]_q dAA\,dA + [2]_q\,[4]_q A(dA)^2 + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q A^2\,d^2A \\ &\quad + [3]_q A\,d^2AA + [2]_q A\,dAA^2 + [3]_q A^2\,dAA \\ &\quad + dAA^3 + [4]_q A^3\,dA + A^5. \end{aligned}$$

Let $\mathfrak{A}$ be a unital associative $\mathbb{C}$-algebra and $\Omega$ be a differential calculus over $\mathfrak{A}$, i.e. $\Omega$ is a graded differential algebra $\Omega = \oplus_k \Omega^k$ with $\Omega^0 = \mathfrak{A}$ and differential $d$. Let $\mathscr{E}$ be a left module over algebra $\mathfrak{A}$. It is evident that $\mathscr{E}$ has the structure of $\mathbb{C}$-vector space induced by a left $\mathfrak{A}$-module structure if one defines $\alpha\xi = (\alpha e)\xi$, where $\alpha \in \mathbb{C}, \xi \in \mathscr{E}$, $e$ is the identity element of algebra $\mathfrak{A}$. Let us remind [9] that an $\Omega$-connection on module $\mathscr{E}$ is a linear map $\nabla : \mathscr{E} \longrightarrow \Omega^1 \otimes_{\mathfrak{A}} \mathscr{E}$ satisfying the condition

$$\nabla(\omega\xi) = d\omega \otimes_{\mathfrak{A}} \xi + \omega\nabla(\xi),$$

where $\omega \in \mathfrak{A}$, $\xi \in \mathscr{E}$. Since $\Omega^k$ can be viewed as the $(\mathfrak{A},\mathfrak{A})$-bimodule, the tensor product $\Omega^1 \otimes_{\mathfrak{A}} \mathscr{E}$ has the structure of the left $\mathfrak{A}$-module. Let us denote $\mathscr{F} = \Omega \otimes_{\mathfrak{A}} \mathscr{E}$. Obviously, $\mathscr{F}$ is the left $\Omega$-module and also a graded left $\mathfrak{A}$-module, i.e. $\mathscr{F} = \oplus_k \mathscr{F}^k$, where $\mathscr{F}^k = \Omega^k \otimes_{\mathfrak{A}} \mathscr{E}$. One can extend an $\Omega$-connection $\nabla$ to any $\Omega^k \otimes_{\mathfrak{A}} \mathscr{E}$ by means of the formula

$$\nabla(\omega \otimes_{\mathfrak{A}} \xi) = d\omega \otimes_{\mathfrak{A}} \xi + (-1)^k \omega\nabla(\xi),$$

where $\omega \in \Omega^k$, $\xi \in \mathscr{E}$.

Let $\Omega_q$ be a graded $q$-differential algebra with $N$-differential $d$. In order to generalize the notion of $\Omega$-connection, we define as in [2–4] an $\Omega_q$-connection $\nabla_q$ on the left $\Omega_q$-module $\Omega_q \otimes_{\mathfrak{A}} \mathscr{E}$ as a linear operator of degree one satisfying the condition

$$\nabla_q(\omega \otimes_{\mathfrak{A}} \xi) = d\omega \otimes_{\mathfrak{A}} \xi + q^{|\omega|}\omega\nabla_q(\xi),$$

where $\omega \in \Omega_q^k$, $\xi \in \mathscr{E}$, and $|\omega|$ is the grading of the homogeneous element of algebra $\Omega_q$. Analogously, if $\mathscr{G}$ is a right $\mathfrak{A}$-module, we define an $\Omega_q$-connection on $\mathscr{G}$ as a linear map $\nabla_q : \mathscr{G} \longrightarrow \mathscr{G} \otimes_{\mathfrak{A}} \Omega_q^1$ such that $\nabla_q(\xi f) = \xi \otimes_{\mathfrak{A}} df + \nabla_q(\xi) f$ for any $\xi \in \mathscr{G}$, $f \in \mathfrak{A}$.

The tensor product $\mathscr{F} = \Omega_q^k \otimes_{\mathbb{C}} \mathscr{E}$ of vector spaces is the graded $\mathbb{C}$-vector space. Let us denote the vector space of linear operators on $\mathscr{F}$ by $\mathfrak{L}(\mathscr{F})$. The graded structure of the vector space $\mathscr{F}$ induces a graduation on the vector space $\mathfrak{L}(\mathscr{F}) = \oplus_k \mathfrak{L}^k(\mathscr{F})$. If $A : \mathscr{F} \longrightarrow \mathscr{F}$ is a homogeneous linear operator, we can extend it to the linear operator $L_A : \mathfrak{L}(\mathscr{F}) \longrightarrow \mathfrak{L}(\mathscr{F})$ on the graded algebra of linear operators $\mathfrak{L}(\mathscr{F})$ by means of the graded $q$-commutator as follows:

$$L_A(B) = [A,B]_q = A \cdot B - q^{|A||B|} B \cdot A,$$

where $B$ is a homogeneous linear operator and $A \cdot B$ is the product of two linear operators. It can be shown that the $N$th power of any $\Omega_q$-connection $\nabla_q$ is the endomorphism of degree $N$ of the left $\Omega_q$-module $\mathscr{F}$. The proof is based on the formula

$$\nabla_q^k(\omega \otimes_{\mathfrak{A}} \xi) = \sum_{m=0}^{k} q^{m|\omega|} \left[ \begin{array}{c} k \\ m \end{array} \right]_q d^{k-m}\omega \, \nabla_q^m(\xi),$$

where $\omega \in \Omega_q$, $\xi \in \mathscr{E}$. This allows us to define the curvature of an $\Omega_q$-connection $\nabla_q$ as the endomorphism $F = \nabla_q^N$ of degree $N$ of the left $\Omega$-module $\mathscr{F}$.

Let $\mathscr{E}$ be a left $\mathfrak{A}$-module. The set of all homomorphisms of $\mathscr{E}$ into $\mathfrak{A}$ has the structure of the dual module of the left $\mathfrak{A}$-module $\mathscr{E}$, and is denoted by $\mathscr{E}^*$. It is easy to see that $\mathscr{E}^*$ is a right $\mathfrak{A}$-module. If $\nabla_q$ is an $\Omega_q$-connection on $\mathscr{E}$, then a linear map $\nabla_q^* : \mathscr{E}^* \longrightarrow \mathscr{E}^* \otimes_{\mathfrak{A}} \Omega_q^1$ defined as

$$\nabla_q^*(\eta)(\xi) = d(\eta(\xi)) - \eta(\nabla_q(\xi)),$$

where $\xi \in \mathscr{E}$, $\eta \in \mathscr{E}^*$, is an $\Omega_q$-connection on the right module $\mathscr{E}^*$. Indeed, for any $f \in \mathfrak{A}$, $\eta \in \mathscr{E}^*$, $\xi \in \mathscr{E}$, we have

$$\begin{aligned} \nabla_q^*(\eta f)(\xi) &= d(\eta f(\xi)) - (\eta f)(\nabla_q \xi) = d(\eta(\xi) f) - \eta(\nabla_q \xi) f \\ &= d(\eta(\xi)) f + \eta(\xi) \otimes_{\mathfrak{A}} df - \eta(\nabla_q \xi) f = \eta(\xi) \otimes_{\mathfrak{A}} df + \nabla_q^*(\eta(\xi)) f. \end{aligned}$$

In order to define a Hermitian structure on a right $\mathfrak{A}$-module $\mathscr{E}$, we assume $\mathfrak{A}$ to be a graded $q$-differential algebra with involution $*$ such that the largest linear subset contained in the convex cone $C \in \mathfrak{A}$ generated by $a^*a$ is equal to zero, i.e. $C \cap (-C) = 0$. The right $\mathfrak{A}$-module $\mathscr{E}$ is called a Hermitian module if $\mathscr{E}$ is endowed with a sesquilinear map $h : \mathscr{E} \times \mathscr{E} \longrightarrow \mathfrak{A}$ which satisfies

$$\begin{aligned} &h(\xi\omega, \xi\omega') = \omega^* h(\xi, \xi')\omega', \quad \forall \omega, \omega' \in \mathfrak{A}, \ \forall \xi, \xi' \in \mathscr{E}, \\ &h(\xi, \xi) \in C, \quad \forall \xi \in \mathscr{E}, \ \text{and} \ h(\xi, \xi) = 0 \Rightarrow \xi = 0. \end{aligned}$$

We have used the convention for a sesquilinear map to take the second argument to be linear. If $\mathscr{E}$ is a Hermitian right $\mathfrak{A}$-module, an $\Omega_q$-connection $\nabla_q$ on $\mathscr{E}$ is said to be consistent with a Hermitian structure of $\mathscr{E}$ if it satisfies

$$dh(\xi, \xi') = h(\nabla_q(\xi), \xi') + h(\xi, \nabla_q(\xi')),$$

where $\xi, \xi' \in \mathscr{E}$.

In analogy with the theory of $\Omega$-connection [9] we can prove that there is an $\Omega_q$-connection on every projective module. For this we need the following proposition.

**Proposition 4.5.** *If $\mathscr{E} = \mathfrak{A} \otimes V$ is a free $\mathfrak{A}$-module, where $V$ is a $\mathbb{C}$-vector space, then $\nabla_q = d \otimes I_V$ is an $\Omega_q$-connection on $\mathscr{E}$ and this connection is flat, i.e. its curvature vanishes.*

*Proof.* Indeed, $\nabla_q : \mathfrak{A} \otimes V \longrightarrow \Omega_q^1 \otimes (\mathfrak{A} \otimes V)$ and

$$\begin{aligned} \nabla_q(f(g \otimes v)) &= (d \otimes I_V)(f(g \otimes v)) = d(fg) \otimes v \\ &= (dfg) \otimes v + f(dg \otimes v) = df \otimes_{\mathfrak{A}} (g \otimes v) + f \nabla_q(g \otimes v), \end{aligned}$$

where $f, g \in \mathfrak{A}$, $v \in V$. As $d^N = 0$ and $q$ is the primitive $N$th root of unity, we get

$$\nabla_q^N(f(g \otimes v)) = \sum_{k+m=N} \left[ \begin{array}{c} N \\ m \end{array} \right]_q d^k f(d^m g \otimes v) = 0,$$

i.e. the curvature of such an $\Omega_q$-connection vanishes. $\square$

**Theorem 4.6.** *Every projective module admits an $\Omega_q$-connection.*

*Proof.* Let $\mathscr{M}$ be a projective module. From the theory of modules it is known that a module $\mathscr{M}$ is projective if and only if there exists a module $\mathscr{N}$ such that $\mathscr{E} = \mathscr{M} \oplus \mathscr{N}$ is a free module. It is well known that a free left $\mathfrak{A}$-module $\mathscr{E}$ can be represented as the tensor product $\mathfrak{A} \otimes V$, where $V$ is a $\mathbb{C}$-vector space. A linear map $\nabla_q = \pi \circ (d \otimes I_V) : \mathscr{M} \longrightarrow \Omega^1_q \otimes_{\mathfrak{A}} \mathscr{M}$ is an $\Omega_q$-connection on a projective module $\mathscr{M}$, where $d \otimes I_V$ is an $\Omega_q$-connection on a left $\mathfrak{A}$-module $\mathscr{E}$, $\pi$ is the projection on the first summand in the direct sum $\mathscr{M} \oplus \mathscr{N}$, and $\pi(\omega \otimes_{\mathfrak{A}} (g \otimes v)) = \omega \otimes_{\mathfrak{A}} \pi(g \otimes v) = \omega \otimes_{\mathfrak{A}} m$, where $\omega \in \Omega^1_q$, $g \in \mathfrak{A}$, $v \in V$, $m \in \mathscr{M}$. Taking into account Proposition 4.5, we get

$$\begin{aligned} \nabla_q(fm) &= \pi((d \otimes I_V)(fm)) = \pi(df \otimes_{\mathfrak{A}} m + fdm) \\ &= df \otimes_{\mathfrak{A}} \pi(m) + f\nabla_q(m) = df \otimes_{\mathfrak{A}} m + f\nabla_q(m), \end{aligned}$$

where $f \in \mathfrak{A}$, $m \in \mathscr{M}$. □

## ACKNOWLEDGEMENT

We gratefully acknowledge the financial support of the Estonian Science Foundation under the research grant ETF 7427.

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## Gradueeritud $q$-diferentsiaalalgebrale tuginev seostuse üldistus

Viktor Abramov ja Olga Liivapuu

On sisse toodud seostuse vormi üldistus, mis tugineb gradueeritud $q$-diferentsiaalalgebrale, kus $q$ on $N$-astme algjuur ühest, ja välja töötatud kõveruse $N$-vormi mõiste. On tõestatud Bianchi samasus kõveruse $N$-vormi jaoks. On uuritud $\Omega_q$-seostust moodulil ja tõestatud, et igal projektiivsel moodulil eksisteerib $\Omega_q$-seostus. On defineeritud Hermite'i struktuuriga kooskõlaline $\Omega_q$-seostus, kui moodulil on antud Hermite'i struktuur.