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MECHANICS

On modelling wave motion in microstructured solids

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Abstract. The Mindlin-type model is used for describing the longitudinal deformation waves in microstructured solids. The evolution equation (one-wave equation) is derived for the hierarchical governing equation (two-wave equation) in the nonlinear case using the asymptotic (reductive perturbation) method. The evolution equation is integrated numerically under harmonic as well as localized initial conditions making use of the pseudospectral method. Analysis of the results demonstrates that the derived evolution equation is able to grasp essential effects of microinertia and elasticity of a microstructure. The influence of these effects can result in the emergence of asymmetric solitary waves.

Key words: nonlinear wave motion, microstructure, hierarchy of waves, evolution equations.

1. INTRODUCTION

In general terms, macrobehaviour of materials depends on properties of the material structure. This is extremely important in contemporary materials science where functionally graded materials, alloys, ceramics, composites, granular materials, etc. are widely used. Proper modelling brings in the scales and hierarchies [6], and the conventional theory of continuous homogeneous media should be considerably enlarged [2,4,11]. The scale dependence involves dispersive effects as shown already in [19]. The hierarchical behaviour in the Whitham sense means that, depending on the ratio of wave characteristics (wavelength) to scales in the material (characteristic scale of a microstructure), the weight of wave operators will be shifted from one to another [21].

One of the ideas to describe the effects of the microstructure is based on Mindlin's model [12]. This model has recently been extensively studied [2,3], mostly in the 1D setting which explicitly explains the main features of the process. It has been shown that such modelling describes well the influence of the microstructure on dispersion and the existence of hierarchies [2,3]. The model permits, for example, understanding the emergence of solitary waves in microstructured materials, both analytically [9] and numerically [17,18]. In addition, there is a wide area of possible applications in nondestructive testing by solving the corresponding inverse problem for determining the material properties [8,10].

Our final interest is to analyse 2D problems. However, a common approach when solving multi-dimensional hyperbolic problems is to apply dimensional splitting, i.e., to iterate on 1D problems and to understand the accuracy of possible approximations.

The model equation in the studies mentioned above is in the 1D case a typical hierarchical wave equation with the leading operator of the 2nd order and the higher-order operators (4th, 6th orders) describing the influence of the microstructure [2,3]. This is the two-wave equation, i.e., it describes waves propagating in two directions. The powerful analytic methods [20] show explicitly how in this case evolution equations that govern the propagation of one wave only could be derived. The best example of such an evolution equation is the celebrated Korteweg–de Vries (KdV) equation. The evolution equations may also include hierarchies like in granular materials [7]. If we are interested in wave propagation along a certain coordinate without

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reflection from boundaries, then the concept of evolution equations is preferable. However, the transformations from a two-wave model to an evolution equation should bring over all the essential features that could influence the velocities or the distortions of the wave profile. It is of great interest to understand how the hierarchies in basic Mindlin-type models are reflected in the corresponding evolution equations and how the solutions describe the dispersive effects. It must be stressed that once we use nonlinear models, the balance between nonlinearity and dispersion is of interest.

The main goals of the present paper are (i) to derive the evolution equation that governs one-wave propagation for Mindlin's model; (ii) to find numerical solutions to the evolution equation, and (iii) to compare the results with those of the two-wave equation.

2. BASIC MODEL AND THE EVOLUTION EQUATION

One-dimensional wave propagation in a microstructured material has been studied by Engelbrecht et al. [1–3] on the basis of Mindlin's model [12], augmented by nonlinear terms. The motion is described by two scalar functions, the macrodisplacement $u(x,t)$ and the microdeformation $\varphi(x,t)$, both depending on the material coordinate x and time t. The functions u and φ are governed by two coupled partial differential equations of the form

$$\begin{aligned} \rho u_{tt} &= a u_{xx} + A\varphi_x + \frac{1}{2}N\left(u_x^2\right)_x, \\ I\varphi_{tt} &= C\varphi_{xx} - Au_x - B\varphi + \frac{1}{2}M\left(\varphi_x^2\right)_x, \end{aligned} \tag{2.1}$$

where ρ and I denote the macrodensity and the microinertia, respectively, and the constants a, A, B, C, N, and M are material parameters specifying the strain energy function. The last two constants, N and M, are responsible for nonlinear effects on the macro- and microscale, respectively.

The main interest is focused on longitudinal waves modified by the presence of the microstructure. For this purpose a single partial differential equation is extracted from the system (2.1), which describes a motion in which the macrodisplacement prevails and the influence of the microstructure is retained in a first approximation. The so-called 'slaving principle' is explained in detail in papers [1–3]. A modified version leading to the same result is presented in [14]. By keeping the original variables and parameters, the resulting equation has the form

$$\begin{aligned} \rho u_{tt} = \left(a - \frac{A^2}{B}\right) u_{xx} + \frac{1}{2}N\left(u_x^2\right)_x + \frac{A^2}{B^2}\left(Iu_{tt} - Cu_{xx}\right)_{xx} \\ + \frac{1}{2}M\frac{A^3}{B^3}\left(u_{xx}^2\right)_{xx}. \end{aligned} \tag{2.2}$$

It is an *approximate* equation extracted from the original system (2.1) by means of the slaving principle.

Equation (2.2) can still be condensed by introducing normalized variables and parameters. First, a reference length l is chosen. From the original material constants an inherent length can be extracted, which represents the size of the microstructure. It is considered to be small compared to the reference length l and is introduced by

$$(\delta l)^2 = \frac{IA^2}{\rho B^2}, \tag{2.3}$$

where the small number $\delta \ll 1$ characterizes the smallness of the microstructure. In addition, the characteristic velocities c, c_1, c_N, and c_M are defined by

$$c^2 = \frac{1}{\rho}\left(a - \frac{A^2}{B}\right), \quad c_1^2 = \frac{C}{I}, \quad c_N^2 = \frac{N}{\rho}, \quad c_M^2 = \frac{MA}{IBl} \tag{2.4}$$

in terms of the basic model parameters and, in the case of c_M^2, also of the standard length l.

The original variables x, t, u are finally replaced by nondimensional variables

$$X = \frac{x}{l}, \quad T = \frac{ct}{l}, \quad \varepsilon U = \frac{u}{l}. \tag{2.5}$$

The normalization of the displacement uses another small number $\varepsilon \ll 1$, which emphasizes that the displacement u is small compared to the reference length l. Using the new dimensionless variables, the governing equation (2.2) assumes the form

$$\begin{aligned} U_{TT} = U_{XX} + \frac{1}{2}\varepsilon\frac{c_N^2}{c^2}\left(U_X^2\right)_X \\ + \delta^2\left(U_{TT} - \frac{c_1^2}{c^2}U_{XX} + \frac{1}{2}\varepsilon\frac{c_M^2}{c^2}U_{XX}^2\right)_{XX}. \end{aligned} \tag{2.6}$$

If omitting dispersive and nonlinear terms in the governing equation (2.6), a simple wave equation would remain, whose general solution would be a left- or right-going wave of arbitrary shape travelling undisturbed. Due to the normalization, their speed would be unity. Let us concentrate on waves propagating to the right. To include the influence of the additional terms of the governing equation, we allow the wave profile to change slowly in time.

In selecting a right-going wave, the solution of the evolution equation is assumed in the form as suggested in [13, p. 6]:

$$U = f(\xi,\tau), \quad \xi = X - T, \quad \tau = \frac{1}{2}\varepsilon T, \tag{2.7}$$

where ξ and τ denote moving space and time coordinates, respectively. Inserting this ansatz into the recent form of the governing equation (2.6) and discarding the higher-order terms, one obtains the equation

$$-f_{\xi\tau} = \frac{c_N^2}{2c^2}\left(f_\xi^2\right)_\xi + \frac{\delta^2}{\varepsilon}\left(f_{\xi\xi} - \frac{c_1^2}{c^2}f_{\xi\xi} + \frac{1}{2}\varepsilon\frac{c_M^2}{c^2}f_{\xi\xi}^2\right)_{\xi\xi}. \tag{2.8}$$

Evidently, the influences of dispersion and macro-nonlinearity, controlled by the two small parameters δ and ε, are balanced only if the quotient δ^2/ε is of the order of unity. Without loss of generality we may assume that ε is equal to δ^2.

If we denote $f_\xi = \alpha$, the evolution equation assumes the form

$$\alpha_\tau + q\left(\alpha^2\right)_\xi + z\alpha_{\xi\xi\xi} + w\left(\alpha_\xi^2\right)_{\xi\xi} = 0, \qquad (2.9)$$

where the parameters

$$q = \frac{c_N^2}{2c^2}, \qquad z = \frac{c^2 - c_1^2}{c^2}, \qquad w = \varepsilon\frac{c_M^2}{2c^2} \qquad (2.10)$$

characterize the nonlinearity of macroscale, the dispersion, and the nonlinearity of microscale, respectively. Equalizing the micro-nonlinearity parameter w to zero yields the well-known KdV equation. Thus, compared with the standard KdV equation, equation (2.9) includes an additional complicated term which reflects the nonlinearity on the macroscale.

3. NUMERICAL SIMULATION

The evolution equation (2.9) is solved under harmonic and localized initial conditions

$$\alpha(\xi,0) = \sin\xi, \qquad \alpha(\xi,0) = A_0\,\mathrm{sech}^2\frac{\xi-\xi_0}{\sqrt{12z/A_0}}, \qquad (3.1)$$

respectively, where A_0 is the amplitude, ξ_0 the initial phase-shift, and $\sqrt{12z/A_0}$ the width of the initial pulse. For numerical integration the FFT-based pseudospectral method is used and the periodic boundary conditions are applied [5].

The crucial question is the proper choice of parameters because not much is known about the values of physical constants of Mindlin's model [12]. We choose here the values of parameters comparable with the standard KdV equation which has been studied in detail (see, for example [15,16]). One of the important features of the standard KdV equation is the emergence of a soliton train. The number of solitons in a train depends on the values of q and z. Widely used values are $q = 1$ and $z = 10^{-2.5}$ [15,16]. Then the soliton train develops at $\tau \approx 30$. Another important feature for the KdV equation is the existence of a single stable soliton.

On the basis of the argumentation above, we take here $q = 1$ and vary the other parameters in the following domains: $10^{-2.5} \leq z \leq 1$ and $0 \leq w \leq 1$. The localized initial wave $(3.1)_2$ is the analytical solution for equation (2.9) in the case of $w = 0$, i.e., it represents the KdV soliton.

3.1. Localized initial excitation

Janno and Engelbrecht [9] have shown that for the two-wave equation (2.6) there exists an asymmetric travelling wave solution, i.e., the nonlinearity in microscale leads to asymmetry of the wave profile. Numerical experiments by Salupere et al. [17,18] have demonstrated that in the case of equation (2.6), an initially symmetric localized wave is deformed to an asymmetric wave during propagation. Here we show that the same effect takes place in the case of the evolution equation (2.9).

The evolution of the initial symmetric sech^2 pulse can be traced in Fig. 1. It is clear that the shape of the wave is altered during propagation and an oscillating tail is formed. In Fig. 2 the initial wave profile and the altered shape of the wave profile at the end of the integration interval are plotted against ξ. In order to characterize the asymmetry of the last wave profile more explicitly, α_ξ is plotted against α in Fig. 3.

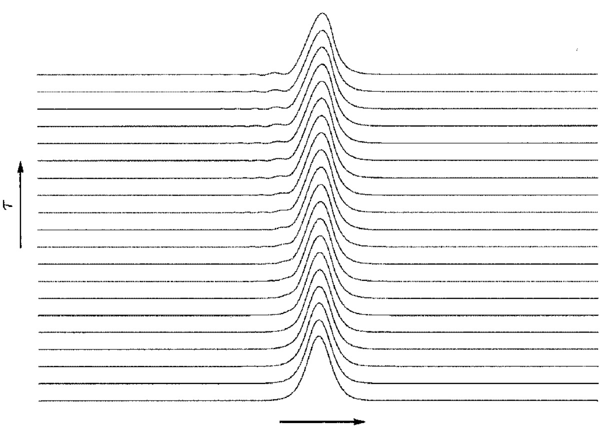

Fig. 1. Time-slice plot for $z = 10^{-2}$, $w = 10^{-2.5}$.

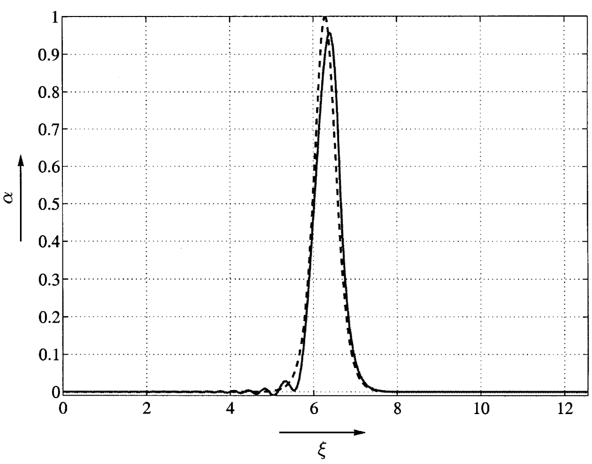

Fig. 2. The initial (dashed line) and the deformed (solid line) wave profile from Fig. 1 ($z = 10^{-2}$, $w = 10^{-2.5}$).

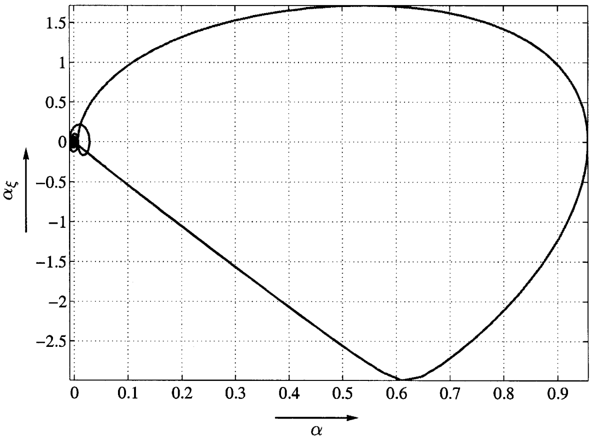

Fig. 3. Asymmetry of the wave profile at the end of the integration interval: α_ξ against α for $z = 10^{-2}$, $w = 10^{-2.5}$.

In applying localized initial conditions the value of the micro-nonlinearity parameter $w = 10^{-2.5}$ is chosen quite big compared to the macro-nonlinearity parameter q and the dispersion parameter z in order to demonstrate the effect of asymmetry more clearly.

3.2. Harmonic initial excitation

It is of interest to start with the case $w = 0$ which corresponds to a standard KdV equation. This means that micro-nonlinearity is neglected. As typical of the KdV case, a train of solitons will emerge from a harmonic initial excitation (Fig. 4.) The interaction picture is complicated but solitons preserve their shape and speed over long time intervals. The soliton amplitudes fluctuate in the interval that is dictated by the interaction rules [15,16]. When the micro-nonlinearity is taken into account, the interaction pattern is altered – speeds of solitons are higher than in the KdV case (cf. Figs 4 and 5). Like in the case of localized initial conditions, the emerged solitons (Fig. 6) are asymmetric, as can be observed from the phase plane, i.e., the (α, α_ξ) plot (Fig. 7). This is a clear sign of the influence of

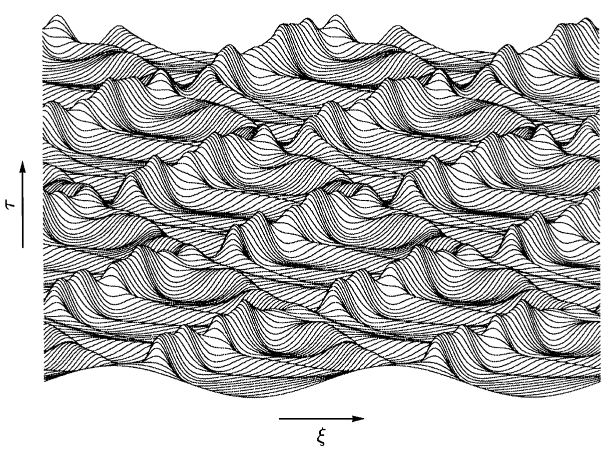

Fig. 4. Time-slice plot over two space periods for the KdV case, $z = 10^{-1.5}$, $w = 0$.

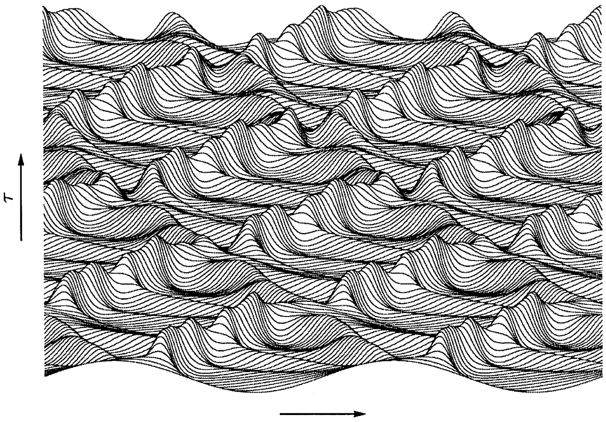

Fig. 5. Time-slice plot over two space periods for $z = 10^{-1.5}$, $w = 10^{-2.621}$.

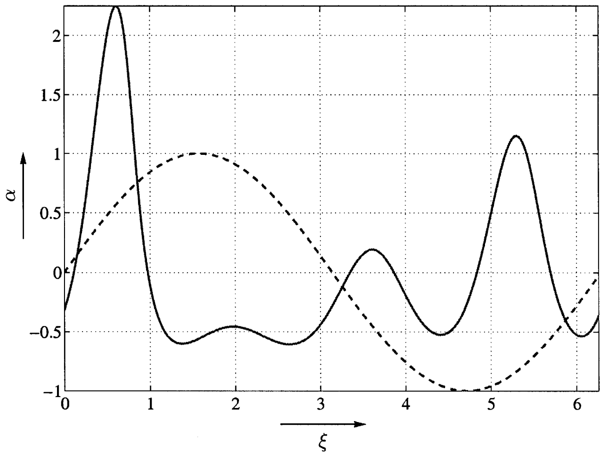

Fig. 6. Initial harmonic wave (dashed line) and wave profile (solid line) at $\tau = 14.3$ for $z = 10^{-1.5}, w = 10^{-2.621}$.

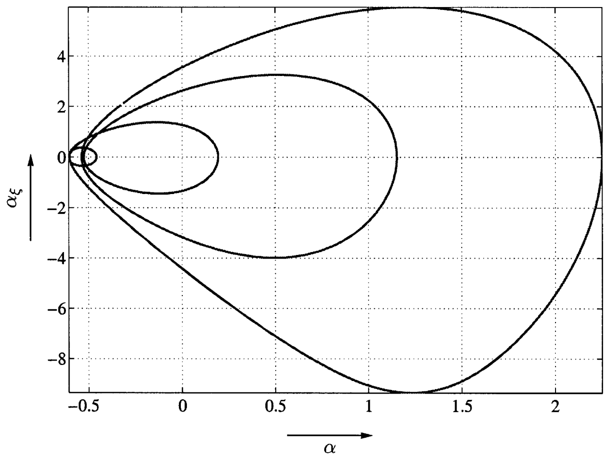

Fig. 7. Asymmetry of solitons: α_ξ against α at $\tau = 14.3$ for $z = 10^{-1.5}, w = 10^{-2.621}$.

micro-nonlinearity. The chosen time instant $\tau = 14.3$ corresponds to the formation of the soliton train at given values of z and w.

4. CONCLUDING REMARKS

The evolution equation (2.9) that governs one-wave propagation in microstructured solids according to Mindlin's model is derived and solved numerically under harmonic and localized initial conditions. Analysis of numerical results demonstrates that (i) for both the governing equation and the evolution equation nonlinearity in microscale leads to asymmetry of the wave profile; and (ii) the stronger the influence of micro-nonlinearity, the more the solutions of the evolution equation differ from those of the KdV model. In conclusion, the derived evolution equation (2.9) – notwithstanding that it is a simplified model equation compared to the two-wave equation (2.6) – is able to grasp essential effects of microinertia and elasticity of a microstructure. However, we stress that the values of parameters used above are chosen for the comparison with the standard KdV equation in order to demonstrate the influence of the microstructure. Studies with other parameters are in progress. A real challenge is to find an analytical solution to equation (2.9).

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Lainelevi modelleerimisest mikrostruktuuriga materjalides

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On tuletatud mikrostruktuuriga materjali (näiteks komposiidid, metallisulamid, granuleeritud materjalid jne) hierarhilise põhivõrrandi jaoks evolutsioonivõrrand ehk nn ühe laine võrrand, mis kirjeldab mittelineaarsust nii makro- kui mikrotasandil, kusjuures dispersiooniefekt on taandatud Kortewegi-de Vriesi tüüpi dispersioonile. See võimaldab kirjeldada laineleviprotsessi piisava füüsikalise täpsusega, jättes kõrvale algse liikumisvõrrandi. Klassikaline KdV-mudel teist järku mittelineaarsuse ja kuupdispersiooniga viib sümmeetrilise üksiklaine tekkeni makrostruktuuri mittelineaarsuse ja dispersiooni tasakaalu korral. Mikrostruktuuri mittelineaarsus aga häirib seda tasakaalu. Nii põhivõrrandit [8–10] kui sellele vastavat evolutsioonivõrrandit on analüüsitud numbriliselt ja näidatud, et mikrostruktuuri mittelineaarsuse tõttu on üksiklaine ebasümmeetriline.