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MATHEMATICS

On weak symmetries of trans-Sasakian manifolds

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Abstract. The present paper deals with weakly symmetric and weakly Ricci symmetric trans-Sasakian manifolds. The existence of weakly Ricci symmetric trans-Sasakian manifolds is ensured by an example.

Key words: local differential geometry, weakly symmetric, weakly Ricci symmetric, α-Sasakian, β-Kenmotsu, trans-Sasakian manifold.

1. INTRODUCTION

As a proper generalization of pseudosymmetric manifolds by Chaki [3], in 1989 Tamássy and Binh [14] introduced the notion of weakly symmetric manifolds. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly symmetric if its curvature tensor $\bar{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\nabla_X \bar{R})(Y,Z,U,V) \;=\;& A(X)\bar{R}(Y,Z,U,V)+B(Y)\bar{R}(X,Z,U,V)\\ &+C(Z)\bar{R}(Y,X,U,V)+D(U)\bar{R}(Y,Z,X,V)\\ &+E(V)\bar{R}(Y,Z,U,X) \qquad (1.1)\end{aligned}$$

for all vector fields $X,Y,Z,U,V \in \chi(M^n)$, where A,B,C,D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. If in (1.1) the 1-form A is replaced by $2A$ and E is replaced by A, then a $(WS)_n$ reduces to the notion of generalized pseudosymmetric manifold by Chaki [4]. In 1999 De and Bandyopadhyay [6] studied a $(WS)_n$ and proved that in such a manifold the associated 1-forms $B = C$ and $D = E$. Hence (1.1) reduces to the following:

$$\begin{aligned}(\nabla_X \bar{R})(Y,Z,U,V) \;=\;& A(X)\bar{R}(Y,Z,U,V)+B(Y)\bar{R}(X,Z,U,V)\\ &+B(Z)\bar{R}(Y,X,U,V)+D(U)\bar{R}(Y,Z,X,V)\\ &+D(V)\bar{R}(Y,Z,U,X). \qquad (1.2)\end{aligned}$$

In 1993 Tamássy and Binh [15] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z)+B(Y)S(X,Z)+C(Z)S(Y,X), \qquad (1.3)$$

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where A, B, C are three non-zero 1-forms, called the associated 1-forms of the manifold, and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(WRS)_n$. As an equivalent notion of $(WRS)_n$, Chaki and Koley [5] introduced the notion of generalized pseudo Ricci symmetric manifold. If in (1.3) the 1-form A is replaced by $2A$ then the definition of $(WRS)_n$ reduces to that of generalized pseudo Ricci symmetric manifold by Chaki and Koley.

Especially, if $A = B = C = 0$, then a $(WRS)_n$ reduces to Ricci-symmetric and if $B = C = 0$, then it reduces to Ricci recurrent.

In 1985, Oubina [11] introduced the notion of trans-Sasakian manifolds, which contains both the class of Sasakian and cosympletic structures, closely related to the locally conformal Kähler manifolds. Trans-Sasakian manifolds of type (0, 0), $(\alpha,0)$, and $(0,\beta)$ are the cosympletic, α-Sasakian, and β-Kenmotsu manifold, respectively. In particular, if $\alpha = 1, \beta = 0$; and $\alpha = 0, \beta = 1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifold, respectively. Thus trans-Sasakian structures provide a large class of generalized quasi-Sasakian structures. In 2002, Kim et al. [10] studied generalized Ricci recurrent trans-Sasakian manifolds. In [9] De and Tripathi studied Ricci semi-symmetric trans-Sasakian manifolds. Trans-Sasakian manifolds were also studied by Shaikh et al. [13].

The object of the present paper is to study weakly symmetric and weakly Ricci symmetric trans-Sasakian manifolds. Section 2 deals with preliminaries of trans-Sasakian manifolds. Tamássy and Binh [15] studied weakly symmetric and weakly Ricci symmetric Sasakian manifolds and proved that in such a manifold the sum of the associated 1-forms vanishes everywhere. Subsequently in [7] De et al. considered weakly symmetric and weakly Ricci symmetric K-contact manifolds. Also De et al. [8] studied weakly symmetric and weakly Ricci symmetric contact metric manifolds with a nullity condition. Again Özgür [12] studied weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds and proved that in such a manifold the sum of the associated 1-forms is zero everywhere and hence such a manifold does not exist unless the sum of the associated 1-forms is everywhere zero. However, in Section 3 of the paper it is proved that the sum of the associated 1-forms of a weakly symmetric trans-Sasakian manifold of non-vanishing ξ-sectional curvature is non-zero everywhere and hence such a structure exists. In Section 4 we study weakly Ricci symmetric trans-Sasakian manifolds and prove that in such a structure, with non-vanishing ξ-sectional curvature, the sum of the associated 1-forms is non-vanishing everywhere and consequently such a structure exists. Finally, Section 5 deals with a concrete example of weakly Ricci symmetric trans-Sasakian manifold that is neither Ricci symmetric nor Ricci-recurrent.

2. TRANS-SASAKIAN MANIFOLDS

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold [1] if it admits a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η, and a Riemannian metric g, which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X,Y) = -g(X,\phi Y), \quad \eta(X) = g(X,\xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X,Y on M.

An almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be trans-Sasakian manifold [11] if $(M\times\mathbb{R},J,G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M\times\mathbb{R}$ defined by

$$J\left(Z,f\frac{d}{dt}\right) = \left(\phi Z - f\xi, \eta(Z)\frac{d}{dt}\right)$$

for any vector field Z on M and smooth function f on $M\times\mathbb{R}$ and G is the product metric on $M\times\mathbb{R}$. This may be stated by the condition [2]

$$(\nabla_X\phi)(Y) = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{2.4}$$

where α, β are smooth functions on M and such a structure is said to be the trans-Sasakian structure of type (α, β). From (2.4) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta\{X - \eta(X)\xi\}, \tag{2.5}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In a trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ the following relations hold [9]:

$$\begin{aligned} R(X,Y)\xi =&(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\phi Y - (X\beta)\phi^2(Y) \\ &+ 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X + (Y\beta)\phi^2(X), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \eta(R(X,Y)Z) =&(\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &- 2\alpha\beta[g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)] \\ &- (Y\alpha)g(\phi X,Z) - (X\beta)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &+ (X\alpha)g(\phi Y,Z) + (Y\beta)\{g(X,Z) - \eta(Z)\eta(X)\}, \end{aligned} \tag{2.8}$$

$$S(X,\xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\phi X)\alpha) - (2n-1)(X\beta), \tag{2.9}$$

$$R(\xi,X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X], \tag{2.10}$$

$$S(\xi,\xi) = 2n(\alpha^2 - \beta^2 - \xi\beta), \tag{2.11}$$

$$(\xi\alpha) + 2\alpha\beta = 0, \tag{2.12}$$

$$Q\xi = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\xi + \phi(\operatorname{grad}\alpha) - (2n-1)(\operatorname{grad}\beta), \tag{2.13}$$

where R is the curvature tensor of type (1, 3) of the manifold and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$ for any vector fields X, Y on M. The ξ-sectional curvature $K(\xi,X) = g(R(\xi,X)\xi,X)$ for a unit vector field X orthogonal to ξ plays an important role in the study of an almost contact metric manifold. Throughout the paper we consider a trans-Sasakian manifold of non-vanishing ξ-sectional curvature.

3. WEAKLY SYMMETRIC TRANS-SASAKIAN MANIFOLDS

Definition 3.1. *A trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is said to be weakly symmetric if its Riemannian curvature tensor* $\bar{R}$ *of type* (0, 4) *satisfies* (1.2).

Let $\{e_i : i = 1, 2, ..., 2n+1\}$ be an orthonormal basis of the tangent space $T_p(M)$ at any point p of the manifold. Then setting $Y = V = e_i$ in (1.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} (\nabla_X S)(Z,U) \quad = \quad & A(X)S(Z,U) + B(Z)S(X,U) + D(U)S(X,Z) \\ & + B(R(X,Z)U) + D(R(X,U)Z). \end{aligned} \tag{3.1}$$

Putting $X = Z = U = \xi$ in (3.1) and then using (2.7) and (2.11) we obtain

$$A(\xi) + B(\xi) + D(\xi) = \frac{2\alpha(\xi\alpha) - 2\beta(\xi\beta) - (\xi(\xi\beta))}{\alpha^2 - (\xi\beta) - \beta^2} \tag{3.2}$$

provided that $\alpha^2 - (\xi\beta) - \beta^2 \neq 0$.

The ξ-sectional curvature $K(\xi,X)$ of a trans-Sasakian manifold for a unit vector field X orthogonal to ξ is given by $K(\xi,X) = g(R(\xi,X)\xi,X)$. Hence (2.10) yields

$$K(\xi,X) = -\{\alpha^2 - (\xi\beta) - \beta^2\}.$$

If $\alpha^2 - (\xi\beta) - \beta^2 = 0$, then the manifold is of vanishing ξ-sectional curvature. Hence we can state the following:

Theorem 3.1. *In a weakly symmetric trans-Sasakian manifold* (M^{2n+1},g) $(n>1)$ *of non-vanishing* ξ*-sectional curvature*, *relation* (3.2) *holds.*

Next, substituting X and Z by ξ in (3.1) and then using (2.9) and (2.12) we obtain

$$(\nabla_\xi S)(\xi,U)=[A(\xi)+B(\xi)]S(U,\xi)+[\alpha^2-(\xi\beta)-\beta^2][(2n-1)D(U)+\eta(U)D(\xi)]. \tag{3.3}$$

Again we have

$$\begin{aligned}(\nabla_\xi S)(\xi,U) &= \nabla_\xi S(\xi,U)-S(\nabla_\xi\xi,U)-S(\xi,\nabla_\xi U)\\ &= \nabla_\xi S(\xi,U)-S(\xi,\nabla_\xi U)\\ &= [2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))]\eta(U)\\ &\quad -(2n-1)(U(\xi\beta))-(\phi U(\xi\alpha)),\end{aligned} \tag{3.4}$$

where (2.9) has been used. In view of (3.3) and (3.4) we obtain from (3.2) that

$$\begin{aligned}D(U) &= \frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))]\eta(U)}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]}\\ &\quad -\frac{(2n-1)(U(\xi\beta))+(\phi U(\xi\alpha))}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]}\\ &\quad +D(\xi)\left[\frac{(2n-1)\{(\alpha^2-\beta^2)\eta(U)-(U\beta)\}-((\phi U)\alpha)}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}}\right]\\ &\quad -\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}^2}[\{2n(\alpha^2-\beta^2)\\ &\quad -(\xi\beta)\}\eta(U)-(2n-1)(U\beta)-((\phi U)\alpha)]\end{aligned} \tag{3.5}$$

for any vector field U, provided that $\alpha^2-(\xi\beta)-\beta^2\neq 0$.

Next, setting $X=U=\xi$ in (3.1) and proceeding in a similar manner as above we get

$$\begin{aligned}B(Z) &= \frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))]\eta(Z)}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]}\\ &\quad -\frac{(2n-1)(Z(\xi\beta))+(\phi Z(\xi\alpha))}{(2n-1)[\alpha^2-(\xi\beta)-\beta^2]}\\ &\quad +B(\xi)\left[\frac{(2n-1)\{(\alpha^2-\beta^2)\eta(Z)-(Z\beta)\}-((\phi Z)\alpha)}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}}\right]\\ &\quad -\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))}{(2n-1)\{\alpha^2-(\xi\beta)-\beta^2\}^2}[\{2n(\alpha^2-\beta^2)\\ &\quad -(\xi\beta)\}\eta(Z)-(2n-1)(Z\beta)-((\phi Z)\alpha)]\end{aligned} \tag{3.6}$$

for any vector field Z, provided that $\alpha^2-(\xi\beta)-\beta^2\neq 0$. This leads to the following:

Theorem 3.2. *In a weakly symmetric trans-Sasakian manifold* (M^{2n+1},g) $(n>1)$ *of non-vanishing* ξ*-sectional curvature*, *the associated* 1*-forms D and B are given by* (3.5) *and* (3.6), *respectively.*

Again, setting $Z = U = \xi$ in (3.1) we get

$$\begin{aligned}(\nabla_X S)(\xi,\xi) &= A(X)S(\xi,\xi)+[B(\xi)+D(\xi)]S(X,\xi)+B(R(X,\xi)\xi)+D(R(X,\xi)\xi)\\ &= 2n[\alpha^2-(\xi\beta)-\beta^2]A(X)+[B(\xi)+D(\xi)]S(X,\xi)\\ &\quad -[\alpha^2-(\xi\beta)-\beta^2][\eta(X)\{B(\xi)+D(\xi)\}-B(X)-D(X)]. \end{aligned}\tag{3.7}$$

Now we have

$$(\nabla_X S)(\xi,\xi) = \nabla_X S(\xi,\xi) - 2S(\nabla_X \xi,\xi),$$

which yields by using (2.5) and (2.9) that

$$\begin{aligned}(\nabla_X S)(\xi,\xi) &= 2n[2\alpha(X\alpha)-2\beta(X\beta)-(X(\xi\beta))]\\ &\quad +2\alpha[(X\alpha)-\eta(X)(\xi\alpha)-(2n-1)((\phi X)\beta)]\\ &\quad +2\beta[((\phi X)\alpha)+(2n-1)\{(X\beta)-(\xi\beta)\eta(X)\}]. \end{aligned}\tag{3.8}$$

In view of (3.5), (3.6), and (3.8), (3.7) yields

$$\begin{aligned}A(X)+B(X)+D(X) &= \frac{2\alpha(X\alpha)-2\beta(X\beta)-(X(\xi\beta))}{\alpha^2-(\xi\beta)-\beta^2}\\ &\quad +\frac{\alpha}{n}\left[\frac{(X\alpha)-\eta(X)(\xi\alpha)-(2n-1)((\phi X)\beta)}{\alpha^2-(\xi\beta)-\beta^2}\right]\\ &\quad +\frac{\beta}{n}\left[\frac{((\phi X)\alpha)+(2n-1)\{(X\beta)-(\xi\beta)\eta(X)\}}{\alpha^2-(\xi\beta)-\beta^2}\right]\\ &\quad -\frac{((\phi X)(\xi\alpha))+(2n-1)(X(\xi\beta))}{n\{\alpha^2-(\xi\beta)-\beta^2\}}\\ &\quad +\frac{[2n\{2\alpha(\xi\alpha)-2\beta(\xi\beta)\}-(\xi(\xi\beta))]\eta(X)}{n\{\alpha^2-(\xi\beta)-\beta^2\}}\\ &\quad -\frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))}{n\{\alpha^2-(\xi\beta)-\beta^2\}^2}[\{2n(\alpha^2-\beta^2)\\ &\quad -(\xi\beta)\}\eta(X)-((\phi X)\alpha)-(2n-1)(X\beta)] \end{aligned}\tag{3.9}$$

for any vector field X, provided that $\alpha^2-(\xi\beta)-\beta^2 \neq 0$. This leads to the following:

Theorem 3.3. *In a weakly symmetric trans-Sasakian manifold* (M^{2n+1},g) $(n>1)$ *of non-vanishing* ξ*-sectional curvature, the sum of the associated* 1*-forms is given by* (3.9).

In particular, if $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$, then $(\xi\beta) = 0$ and hence relation (3.9) reduces to the following form

$$\begin{aligned}A(X)+B(X)+D(X) &= \frac{2\alpha(X\alpha)-2\beta(X\beta)}{\alpha^2-\beta^2}+\frac{\alpha\{(X\alpha)-\eta(X)(\xi\alpha)-(2n-1)((\phi X)\beta)\}}{n(\alpha^2-\beta^2)}\\ &\quad +\frac{\beta\{((\phi X)\alpha)+(2n-1)(X\beta)\}+4n\alpha(\xi\alpha)\eta(X)-(\phi X(\xi\alpha))}{n(\alpha^2-\beta^2)}\\ &\quad -\frac{2\alpha(\xi\alpha)[2n(\alpha^2-\beta^2)\eta(X)-((\phi X)\alpha)-(2n-1)(X\beta)]}{n\{(\alpha^2-\beta^2)\}^2} \end{aligned}\tag{3.10}$$

for any vector field X, provided that $\alpha^2-\beta^2 \neq 0$.

If $\alpha^2-\beta^2 = 0$, then in view of (2.12) it can be easily shown from (3.10) that $\alpha = 0 = \beta$ and hence the manifold is cosympletic. This leads to the following:

Corollary 3.1. *If a weakly symmetric non-cosympletic trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *satisfies the condition* $\phi(\mathrm{grad}\,\alpha) = \mathrm{grad}\,\beta$, *then the sum of the associated* 1*-forms is given by* (3.10).

If $\beta = 0$ and $\alpha = 1$, then (3.9) yields $A(X)+B(X)+D(X) = 0$ for all X and hence we can state the following:

Corollary 3.2 [15]**.** *There is no weakly symmetric Sasakian manifold* $M^{2n+1}(n > 1)$, *unless the sum of the* 1*-forms is everywhere zero.*

Corollary 3.3. *If an* α*-Sasakian manifold is weakly symmetric, then the sum of the* 1*-forms, i.e.* $A+B+D$, *is given by*

$$A(X)+B(X)+D(X) = \frac{(2n+1)(X\alpha)-\eta(X)(\xi\alpha)}{n\alpha} + \frac{2(\xi\alpha)((\phi X)\alpha)-\alpha(\phi X(\xi\alpha))}{n\alpha^3}.$$

Again, if $\alpha = 0$ and $\beta = 1$, then (3.9) yields $A(X)+B(X)+D(X) = 0$ for all X. This leads to the following:

Corollary 3.4 [12]**.** *There is no weakly symmetric Kenmotsu manifold* $M^{2n+1}(n > 1)$, *unless the sum of the* 1*-forms is everywhere zero.*

Corollary 3.5. *If a* β*-Kenmotsu manifold is weakly symmetric, then the sum of the* 1*-forms, i.e.* $A+B+D$, *is given by*

$$\begin{aligned} A(X)+B(X)+D(X) &= \frac{2\beta(X\beta)+(X(\xi\beta))-(2n-1)\beta\{(X\beta)-(\xi\beta)\eta(X)\}}{(\xi\beta)+\beta^2} \\ &+\frac{\{4n\beta(\xi\beta)+(\xi(\xi\beta))\}\eta(X)+(2n-1)(X(\xi\beta))}{n\{(\xi\beta)+\beta^2\}} \\ &-\frac{\{2\beta(\xi\beta)+(\xi(\xi\beta))\}[\{2n\beta^2+(\xi\beta)\}\eta(X)+(2n-1)(X\beta)]}{n\{(\xi\beta)+\beta^2\}^2}. \end{aligned}$$

4. WEAKLY RICCI SYMMETRIC TRANS-SASAKIAN MANIFOLDS

Definition 4.1. *A trans-Sasakian manifold* $M^{2n+1}(\phi, \xi, \eta, g)$ *is said to be weakly Ricci symmetric if its Ricci tensor of type* (0, 2) *is not identically zero and satisfies relation* (1.3).

Theorem 4.1. *In a weakly Ricci symmetric trans-Sasakian manifold* (M^{2n+1}, g) *of non-vanishing* ξ*-sectional curvature the following relations hold:*

$$A(\xi)+B(\xi)+C(\xi) = \frac{2\alpha(\xi\alpha)-2\beta(\xi\beta)-(\xi(\xi\beta))}{\alpha^2-(\xi\beta)-\beta^2}, \tag{4.1}$$

$$\begin{aligned} &[r-2n(\alpha^2-\beta^2)+(\xi\beta)][A(\xi)+B(\xi)] \\ &\quad= \frac{r[2\alpha(\xi\alpha)-3\beta(\xi\beta)-(\xi(\xi\beta))+\beta(\alpha^2-\beta^2)]}{\alpha^2-(\xi\beta)-\beta^2} \\ &\quad+8n\beta\{\alpha^2+(\xi\beta)\}+(\xi(\xi\beta))-2n(2n+1)\beta(\alpha^2-\beta^2)+\mathrm{div}(\phi\,\mathrm{grad}\,\alpha) \\ &\quad+(2n-1)\{\mathrm{div}(\mathrm{grad}\,\beta)-(\rho_1\beta)-(\rho_2\beta)\}-\{((\phi\rho_1)\alpha)+((\phi\rho_2)\alpha)\}-\alpha\psi, \end{aligned} \tag{4.2}$$

where r is the scalar curvature of the manifold, div *denotes the divergence,* ρ_1, ρ_2 *being the associated vector fields corresponding to the* 1*-forms A and B, respectively, and* $\psi = tr(Q\phi)$.

Proof. From (1.3) it follows that

$$(\nabla_X S)(Y,\xi) = A(X)S(Y,\xi) + B(Y)S(X,\xi) + C(\xi)S(Y,X). \tag{4.3}$$

In view of (2.9) we obtain from (4.3)

$$\begin{aligned}
&A(X)[\{2n(\alpha^2-\beta^2)-(\xi\beta)\}\eta(Y)-(2n-1)(Y\beta)-((\phi Y)\alpha)] \\
&\qquad +B(Y)[\{2n(\alpha^2-\beta^2)-(\xi\beta)\}\eta(X)-(2n-1)(X\beta)-((\phi X)\alpha)]+C(\xi)S(X,Y) \\
&\quad = 4n[\alpha(X\alpha)-\beta(X\beta)]\eta(Y)-(X(\xi\beta))\eta(Y) \\
&\qquad -(2n-1)(X(Y\beta))-(X((\phi Y)\alpha))-\alpha[2n(\alpha^2-\beta^2)-(\xi\beta)]g(Y,\phi X) \\
&\qquad +\beta[2n(\alpha^2-\beta^2)-(\xi\beta)]g(Y,X)+\alpha S(\phi X,Y)-\beta S(X,Y) \\
&\qquad +(2n-1)[(\nabla_X Y\beta)-\beta(Y\beta)\eta(X)]-\beta((\phi Y)\alpha)\eta(X),
\end{aligned} \tag{4.4}$$

where (2.11) has been used.

Setting $X = Y = \xi$ in (4.4) and then using (2.11) we obtain relation (4.1). Let $\{e_i : i = 1,2,\cdots,2n+1\}$ be an orthonormal basis of the tangent space T_pM at any point of the manifold. Then setting $X = Y = e_i$ in (4.4) and taking summation over $i, 1 \le i \le 2n+1$ and then using (2.9) and (2.12) we obtain

$$\begin{aligned}
&[A(\xi)+B(\xi)][2n(\alpha^2-\beta^2)-(\xi\beta)]-(2n-1)[(\rho_1\beta)+(\rho_2\beta)]-[((\phi\rho_1)\alpha)+((\phi\rho_2)\alpha)]+rC(\xi) \\
&\quad = -8n\alpha^2\beta-8n\beta(\xi\beta)-(\xi(\xi\beta))+2n(2n+1)\beta(\alpha^2-\beta^2)-(2n-1)\mathrm{div}(\mathrm{grad}\,\beta) \\
&\qquad -\mathrm{div}(\phi\ \mathrm{grad}\,\alpha)+\alpha\psi-\beta r,
\end{aligned} \tag{4.5}$$

where $\psi = \sum_{i=1}^{2n+1} S(\phi e_i, e_i)$. Eliminating $C(\xi)$ from (4.1) and (4.5) we obtain (4.2). This proves the theorem.

Theorem 4.2. *In a weakly Ricci symmetric trans-Sasakian manifold the Ricci tensor is given by the following*:

$$\begin{aligned}
&[\alpha^2+\{\beta+C(\xi)\}^2]S(X,Y) \\
&\quad =[A(X)\{\beta+C(\xi)\}+\alpha A(\phi X)-\alpha^2\eta(X)][\{2n(\alpha^2-\beta^2)-(\xi\beta)\}\eta(Y) \\
&\qquad -(2n-1)(Y\beta)-((\phi Y)\alpha)]+B(Y)\{\beta+C(\xi)\}[\{2n(\alpha^2-\beta^2)-(\xi\beta)\}\eta(X) \\
&\qquad -(2n-1)(X\beta)-((\phi X)\alpha)]+\{\beta+C(\xi)\}[(X(\xi\beta))\eta(Y)+(2n-1)(X(Y\beta)) \\
&\qquad -4n\{\alpha(X\alpha)-\beta(X\beta)\}\eta(Y)+(X((\phi Y)\alpha))-\phi(X(Y\alpha)) \\
&\qquad -(2n-1)(X(Y\beta))+(2n-1)\beta(Y\beta)\eta(X)+\beta((\phi Y)\alpha)\eta(X)] \\
&\qquad +\alpha C(\xi)[2n(\alpha^2-\beta^2)-(\xi\beta)]g(Y,\phi X)-[\beta\{\beta+C(\xi)\}+\alpha^2][2n(\alpha^2-\beta^2)-(\xi\beta)]g(X,Y) \\
&\qquad +\alpha^2[2n(\alpha^2-\beta^2)-(\xi\beta)]\eta(X)\eta(Y)+\alpha[(\phi X(\xi\beta))\eta(Y)-B(Y)\{(2n-1)((\phi X)\beta) \\
&\qquad -(X\alpha)+\eta(X)(\xi\alpha)\}+(2n-1)(\phi X(Y\beta))+(\phi X((\phi Y)\alpha)) \\
&\qquad -4n\{\alpha((\phi X)\alpha)-\beta((\phi X)\beta)\}\eta(Y)-\phi(\phi X(Y\alpha))-(2n-1)(\phi X(Y\beta))].
\end{aligned} \tag{4.6}$$

Proof. In (4.4) replacing X by ϕX and then using (2.1) and (2.9) we obtain

$$\begin{aligned}
[\beta+C(\xi)]S(Y,\phi X) = &-\alpha S(X,Y)+\{\alpha\eta(X)-A(\phi X)\}S(Y,\xi)\\
&+B(Y)\{(2n-1)((\phi X)\beta)+((\phi^2X)\alpha)\}\\
&+4n[\alpha((\phi X)\alpha)-\beta((\phi X)\beta)]\eta(Y)\\
&-[(\phi X(\xi\beta))\eta(Y)+(2n-1)(\phi X(Y\beta))]\\
&-(\phi X((\phi Y)\alpha))\\
&-\alpha[2n(\alpha^2-\beta^2)-(\xi\beta)]g(Y,\phi^2X)\\
&+\beta[2n(\alpha^2-\beta^2)-(\xi\beta)]g(\phi X,Y)\\
&+((\phi\nabla_{\phi X}Y)\alpha)+(2n-1)(\nabla_{\phi X}Y\beta). \qquad (4.7)
\end{aligned}$$

Eliminating $S(Y,\phi X)$ from (4.4) and (4.7) and using (2.1) and (2.9) we get (4.6). Hence the theorem.

5. EXAMPLE OF WEAKLY RICCI SYMMETRIC TRANS-SASAKIAN MANIFOLDS

Example 5.1. We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{E_1,E_2,E_3\}$ be a linearly independent global frame on M given by

$$E_1=e^z\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right),\quad E_2=e^z\frac{\partial}{\partial y},\quad E_3=\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(E_1,E_3)=g(E_2,E_3)=g(E_1,E_2)=0$, $g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1$. Let η be the 1-form defined by $\eta(U)=g(U,E_3)$ for any $U\in\chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1=E_2$, $\phi E_2=-E_1$, $\phi E_3=0$. Then using the linearity of ϕ and g we have $\eta(E_3)=1$, $\phi^2U=-U+\eta(U)E_3$ and $g(\phi U,\phi W)=g(U,W)-\eta(U)\eta(W)$ for any $U,W\in\chi(M)$. Thus for $E_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g of type (1, 3). Then we have

$$[E_1,E_2]=ye^zE_2-e^{2z}E_3,\quad [E_1,E_3]=-E_1,\quad [E_2,E_3]=-E_2.$$

Taking $E_3=\xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{E_1}E_3=-E_1+\frac{1}{2}e^{2z}E_2,\ \ \nabla_{E_3}E_3=0,\ \ \nabla_{E_2}E_3=-E_2-\frac{1}{2}e^{2z}E_1,$$

$$\nabla_{E_2}E_2=E_3+ye^zE_1,\ \ \nabla_{E_1}E_2=-\frac{1}{2}e^{2z}E_3,\ \ \nabla_{E_2}E_1=\frac{1}{2}e^{2z}E_3-ye^zE_2,$$

$$\nabla_{E_1}E_1=E_3,\ \ \nabla_{E_3}E_2=-\frac{1}{2}e^{2z}E_1,\ \ \nabla_{E_3}E_1=\frac{1}{2}e^{2z}E_2.$$

From the above it can be easily seen that (ϕ,ξ,η,g) is a trans-Sasakian structure on M. Consequently $M^3(\phi,\xi,\eta,g)$ is a trans-Sasakian manifold with $\alpha=-\frac{1}{2}e^{2z}\neq 0$ and $\beta=-1$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$R(E_1,E_2)E_2=-\left(\frac{3}{4}e^{4z}+1\right)E_1,\ \ R(E_1,E_2)E_1=\left(\frac{3}{4}e^{4z}+1\right)E_2+ye^{3z}E_3,$$

$$R(E_2,E_3)E_3 = \left(\frac{1}{4}e^{4z}-1\right)E_2,\ \ R(E_1,E_3)E_3 = \left(\frac{1}{4}e^{4z}-1\right)E_1,$$

$$R(E_1,E_3)E_2 = -ye^{3z}E_1,\ \ R(E_1,E_3)E_1 = ye^{3z}E_2+\left(1-\frac{1}{4}e^{4z}\right)E_3,$$

$$R(E_2,E_3)E_2 = \left(1-\frac{1}{4}e^{4z}\right)E_3,\ \ R(E_1,E_2)E_3 = -ye^{3z}E_1,$$

and the components which can be obtained from these by the symmetry properties.

Using the components of the curvature tensor, we can easily calculate the non-vanishing components of the Ricci tensor S and its covariant derivatives as follows:

$$S(E_1,E_1) = -2\left(\frac{1}{4}e^{4z}+1\right),\ \ S(E_2,E_2) = -e^{4z},$$

$$S(E_3,E_3) = 2\left(\frac{1}{4}e^{4z}-1\right),\ \ S(E_2,E_3) = -ye^{3z},$$

$$(\nabla_{E_1}S)(E_1,E_1) = -2ye^{5z},\ (\nabla_{E_1}S)(E_2,E_2) = -5ye^{5z},\ (\nabla_{E_1}S)(E_3,E_3) = 3ye^{5z},$$

$$(\nabla_{E_1}S)(E_1,E_3) = -e^{4z},\ (\nabla_{E_1}S)(E_1,E_2) = ye^{3z},\ (\nabla_{E_1}S)(E_2,E_3) = -3y^2e^{4z}+\frac{3}{4}e^{6z}-e^{2z},$$

$$(\nabla_{E_2}S)(E_2,E_2) = 2ye^{3z},\ (\nabla_{E_2}S)(E_3,E_3) = -2ye^{3z},\ (\nabla_{E_2}S)(E_1,E_3) = -y^2e^{4z}-\frac{1}{2}e^{6z},$$

$$(\nabla_{E_2}S)(E_1,E_2) = 2ye^{z},\ (\nabla_{E_2}S)(E_2,E_3) = 2\left(1-\frac{5}{4}e^{4z}\right),\ (\nabla_{E_3}S)(E_1,E_1) = -2e^{4z},$$

$$(\nabla_{E_3}S)(E_2,E_2) = -4e^{4z},\ (\nabla_{E_3}S)(E_3,E_3) = 2e^{4z},$$

$$(\nabla_{E_3}S)(E_1,E_3) = \frac{1}{2}ye^{5z},\ \ (\nabla_{E_3}S)(E_2,E_3) = -3ye^{3z}.$$

Since $\{E_1, E_2, E_3\}$ is an orthonormal basis of (M^3, g), any vector X and Y can be written as

$$X = a_1E_1+a_2E_2+a_3E_3,\ \ Y = b_1E_1+b_2E_2+b_3E_3,$$

where a_i, b_i $(i = 1,2,3)$ are positive real numbers. Now

$$\begin{aligned} S(X,Y) &= a_1b_1S(E_1,E_1)+a_2b_2S(E_2,E_2)+a_3b_3S(E_3,E_3) \\ &\quad +(a_1b_2+a_2b_1)S(E_1,E_2)+(a_1b_3+a_3b_1)S(E_1,E_3) \\ &\quad +(a_2b_3+a_3b_2)S(E_2,E_3) \\ &= \frac{1}{2}(a_3b_3-a_1b_1-2a_2b_2)e^{4z}-2(a_1b_1+a_3b_3) \\ &\quad -(a_2b_3+a_3b_2)ye^{3z} \\ &= u_1,\ \text{say.} \end{aligned} \tag{5.1}$$

We choose a_i and b_i $(i = 1,2,3)$ in such a way that $S(X,Y) = u_1 \neq 0$. The covariant derivatives of the Ricci tensor $S(X,Y)$ are given by

$$\begin{aligned} (\nabla_{E_1}S)(X,Y) &= (3a_3b_3-2a_1b_1-5a_2b_2)ye^{5z} \\ &\quad +(a_1b_2+a_2b_1)ye^{3z}-(a_1b_3+a_3b_1)e^{4z} \\ &\quad +(a_2b_3+a_3b_2)\left(\frac{3}{4}e^{6z}-3y^2e^{4z}-e^{2z}\right) \\ &= u_2,\ \text{say.} \end{aligned} \tag{5.2}$$

$$
\begin{aligned}
(\nabla_{E_2}S)(X,Y) &= 2(a_2b_2-a_3b_3)ye^{3z}-(a_1b_3+a_3b_1)\left(y^2e^{4z}+\frac{1}{2}e^{6z}\right)\\
&\quad -(2a_2b_3+3a_3b_2)e^{4z}+2(a_1b_2+a_2b_1)ye^{z}+2(a_2b_3+a_3b_2)\\
&= u_3,\ \text{say}, \qquad (5.3)
\end{aligned}
$$

and

$$
\begin{aligned}
(\nabla_{E_3}S)(X,Y) &= 2(a_3b_3-a_1b_1-2a_2b_2)e^{4z}-3(a_2b_3+a_3b_2)ye^{3z}\\
&\quad +(a_1b_2+a_2b_1)\left(\frac{1}{4}e^{6z}-e^{2z}\right)+\frac{1}{2}(a_1b_3+a_3b_1)ye^{5z}\\
&= u_4,\ \text{say}. \qquad (5.4)
\end{aligned}
$$

Now let us consider the 1-forms as follows:

$$
\begin{aligned}
A(E_1) &= \frac{u_2}{u_1},\\
B(E_1) &= (a_2b_3-a_3b_2)e^{4z},\\
C(E_1) &= (a_3b_2-a_2b_3)e^{4z},\\
A(E_2) &= \frac{u_3}{u_1},\\
B(E_2) &= 0,\\
C(E_2) &= 0,\\
A(E_3) &= \frac{u_4}{u_1},\\
B(E_3) &= 0,\\
C(E_3) &= 0.
\end{aligned}
$$

With these 1-forms the manifold under consideration is a weakly Ricci symmetric trans-Sasakian manifold. This leads to the following:

Theorem 5.1. *There exists a trans-Sasakian manifold (M^3,g) which is weakly Ricci symmetric but neither Ricci symmetric nor Ricci-recurrent.*

Remark. Özgür [12] proved that in a weakly Ricci symmetric Kenmotsu manifold the sum of its associated 1-forms is zero everywhere, but in a weakly Ricci symmetric trans-Sasakian manifold the sum of its associated 1-forms is non-zero everywhere unless the manifold is of non-vanishing ξ-sectional curvature.

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Trans-Sasaki muutkondade nõrkadest sümmeetriatest

Absos Ali Shaikh ja Shyamal Kumar Hui

On vaadeldud nõrgalt sümmeetrilisi ja nõrgalt Ricci sümmeetrilisi trans-Sasaki muutkondi. On tõestatud, et juhul kui muutkond on nõrgalt sümmeetriline trans-Sasaki muutkond või nõrgalt Ricci sümmeetriline trans-Sasaki muutkond, siis assotsieeritud 1-vormide summa $A + B + C$ (vt valem 1.1) on nullist erinev muutkonna igas punktis, millest järeldub, et selline struktuur eksisteerib. Alajaotuses 5 on konstrueeritud nõrgalt Ricci sümmeetrilise trans-Sasaki muutkonna konkreetne näide ja näidatud, et konstrueeritud muutkond ei kuulu Ricci sümmeetriliste või Ricci-rekurrentsete muutkondade klassi.