Strong minimizers of the calculus of variations on time scales and the Weierstrass condition

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Abstract. We introduce the notion of strong local minimizer for the problems of the calculus of variations on time scales. Simple examples show that on a time scale a weak minimum is not necessarily a strong minimum. A time scale form of the Weierstrass necessary optimality condition is proved, which enables to include and generalize in the same result both continuous-time and discrete-time conditions.

Key words: calculus of variations, time scales, strong minimizers, Weierstrass optimality condition.

1. INTRODUCTION

Dynamic equations on time scales are a recent subject that allow the unification and extension of the study of differential and difference equations in one and the same theory [10].

The calculus of variations on time scales was introduced in 2004 with the papers of Bohner [6] and Hilscher and Zeidan [15]. Roughly speaking, in [6] the basic problem of the calculus of variations on time scales with given boundary conditions is introduced, and time scale versions of the classical necessary optimality conditions of Euler-Lagrange and Legendre proved, while in [15] necessary conditions as well as sufficient conditions for variable end-points calculus of variations problems on time scales are established. Since the two pioneer works [6,15] and the understanding that much remains to be done in the area [13], several recent studies have been dedicated to the calculus of variations on time scales: the time scale Euler–Lagrange equation was proved for problems with double delta-integrals [9] and for problems with higher-order delta-derivatives [14]; a correspondence between the existence of variational symmetries and the existence of conserved quantities along the respective Euler–Lagrange delta-extremals was established in [5]; optimality conditions for isoperimetric problems on time scales with multiple constraints and Pareto optimality conditions for multiobjective delta variational problems were studied in [20]; a weak maximum principle for optimal control problems on time scales was obtained in [16]. Such results may also be formulated via the nabla-calculus on time scales, and seem to have interesting applications in economics [1–3,21].

In all the works available in the literature on time scales the variational extrema are regarded in a weak local sense. Differently, here we consider strong solutions of problems of the calculus of variations on time scales. In Section 2 we briefly review the necessary results of the calculus on time scales. The reader

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interested in the theory of time scales is referred to [10,11], while for the classical continuous-time calculus of variations we refer to [12,19], and for the discrete-time setting to [18]. In Section 3 the concept of strong local minimum is introduced (cf. Definition 3.1), and an example of a problem of the calculus of variations on the time scale $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is considered showing that the standard weak minimum used in the literature on time scales is not necessarily a strong minimum (cf. Example 3.2). Our main result is a time scale version of the Weierstrass necessary optimality condition for strong local minimum (cf. Theorem 3.3). We end with Section 4, illustrating our main result with the particular cases of discrete-time and q-calculus of variations [4].

2. TIME SCALES CALCULUS

In this section we introduce basic definitions and results that will be needed for the rest of the paper. For a more general theory of calculus on time scales, we refer the reader to [10,11].

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and it is denoted by $\mathbb{T}$. Thus, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ are trivial examples of time scales. Other examples of time scales are: $[-2,4] \bigcup \mathbb{N}$, $h\mathbb{Z} := \{hz | z \in \mathbb{Z}\}$ for some $h > 0$, $q^{\mathbb{N}_0} := \{q^k | k \in \mathbb{N}_0\}$ for some $q > 1$, and the Cantor set. We assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T},$$

with $\inf\emptyset = \sup\mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. The set $\mathbb{T}^\kappa$ is defined as $\mathbb{T}$ without the left-scattered maximum of $\mathbb{T}$ (in case it exists).

The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$

Example 2.1. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\rho(t) = t-1$, and $\mu(t) = 1$. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a fixed real number, then we have $\sigma(t) = qt$, $\rho(t) = q^{-1}t$, and $\mu(t) = (q-1)t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is *regulated* if the right-hand limit $f(t+)$ exists (finite) at all right-dense points $t \in \mathbb{T}$ and the left-hand limit $f(t-)$ exists at all left-dense points $t \in \mathbb{T}$. A function f is *rd-continuous* (we write $f \in C_{rd}$) if it is regulated and if it is continuous at all right-dense points $t \in \mathbb{T}$. Following [15], a function f is *piecewise rd-continuous* (we write $f \in C_{prd}$) if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\triangle(t)$ such that for all $\varepsilon > 0$ there is a neighbourhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\triangle(t)$ the *delta derivative* of f at t and say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^\kappa$. Note that in right-dense points $f^\triangle(t) = \lim_{s \to t} = \frac{f(t) - f(s)}{t - s}$ provided this limit exists, and in right-scattered points $f^\triangle(t) = \frac{f(\sigma(t) - f(t))}{\mu(t)}$ provided f is continuous at t.

Example 2.2. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$, i.e., the delta derivative coincides with the usual one. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}$, i.e., we get the usual derivative of quantum calculus [17].

Let $f, g : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^\kappa$. Then (see, e.g., [10]),

(i) the product fg is delta differentiable at t with

$$(fg)^\triangle(t) = f^\triangle(t)g^\sigma(t) + f(t)g^\triangle(t) = f^\triangle(t)g(t) + f^\sigma(t)g^\triangle(t);$$

(ii) if $g(t)g^\sigma(t) \neq 0$, then $\frac{f}{g}$ is delta differentiable at t with

$$\left(\frac{f}{g}\right)^\triangle(t) = \frac{f^\triangle(t)g(t) - f(t)g^\triangle(t)}{g(t)g^\sigma(t)},$$

where we abbreviate here and throughout the text $f \circ \sigma$ by f^σ.

A function f is *rd-continuously delta differentiable* (we write $f \in C^1_{rd}$) if $f^\triangle$ exists for all $t \in \mathbb{T}^\kappa$ and $f^\triangle \in C_{rd}$. A continuous function f is *piecewise rd-continuously delta differentiable* (we write $f \in C^1_{prd}$) if f is continuous and $f^\triangle$ exists for all, except possibly at finitely many, $t \in \mathbb{T}^\kappa$ and $f^\triangle \in C_{rd}$. It is known that piecewise rd-continuous functions possess an *antiderivative*, i.e., there exists a function F with $F^\triangle = f$, and in this case the *delta integral* is defined by $\int_c^d f(t)\triangle t = F(d) - F(c)$ for all $c, d \in \mathbb{T}$.

Example 2.3. Let $a, b \in \mathbb{T}$ with $a < b$. If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the classical Riemann integral. If $\mathbb{T} = \mathbb{Z}$, then $\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k)$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $\int_a^b f(t)\Delta t = (1-q)\sum_{t \in [a,b)} t f(t)$.

The delta integral has the following properties (see, e.g., [10]):

(i) if $f \in C_{prd}$ and $t \in \mathbb{T}^\kappa$, then

$$\int_t^{\sigma(t)} f(\tau)\triangle\tau = \mu(t)f(t);$$

(ii) if $c, d \in \mathbb{T}$ and $f, g \in C_{prd}$, then

$$\int_c^d f(\sigma(t))g^\triangle(t)\triangle t = [(fg)(t)]_{t=c}^{t=d} - \int_c^d f^\triangle(t)g(t)\triangle t;$$

$$\int_c^d f(t)g^\triangle(t)\triangle t = [(fg)(t)]_{t=c}^{t=d} - \int_c^d f^\triangle(t)g(\sigma(t))\triangle t.$$

3. THE WEIERSTRASS NECESSARY CONDITION

Let $\mathbb{T}$ be a bounded time scale. Throughout we let $t_0, t_1 \in \mathbb{T}$ with $t_0 < t_1$. For an interval $[t_0, t_1] \cap \mathbb{T}$ we simply write $[t_0, t_1]$. The problem of the calculus of variations on time scales under consideration has the form

$$\text{minimize} \quad \mathscr{L}[x] = \int_{t_0}^{t_1} f(t, x^\sigma(t), x^\triangle(t))\triangle t \tag{3.1}$$

over all $x \in C^1_{prd}$ satisfying the boundary conditions

$$x(t_0) = \alpha, \; x(t_1) = \beta, \quad \alpha, \beta \in \mathbb{R}, \tag{3.2}$$

where $f : [t_0, t_1]^\kappa \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

A function $x \in C^1_{prd}$ is said to be admissible if it satisfies conditions (3.2).

Let us consider two norms in C^1_{prd}:

$$\|x\|_1 = \sup_{t \in [t_0,t_1]^\kappa} |x^\sigma(t)| + \sup_{t \in [t_0,t_1]^\kappa \setminus T} |x^\triangle(t)|,$$

where here and subsequently T denotes the set of points of $[t_0,t_1]^\kappa$ where $x^\triangle(t)$ does not exist, and

$$\|x\|_0 = \sup_{t \in [t_0,t_1]^\kappa} |x^\sigma(t)|.$$

The norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are called the *strong* and the *weak* norm, respectively. The strong and weak norms lead to the following definitions for local minimum:

Definition 3.1. *An admissible function $\bar{x}$ is said to be a* strong local minimum *for (3.1)–(3.2) if there exists $\delta > 0$ such that $\mathscr{L}[\bar{x}] \leq \mathscr{L}[x]$ for all admissible x with $\|x - \bar{x}\|_0 < \delta$. Likewise, an admissible function $\bar{x}$ is said to be a* weak local minimum *for (3.1)–(3.2) if there exists $\delta > 0$ such that $\mathscr{L}[\bar{x}] \leq \mathscr{L}[x]$ for all admissible x with $\|x - \bar{x}\|_1 < \delta$.*

A weak minimum need not necessarily be a strong minimum:

Example 3.2. Consider the variational problem

$$\mathscr{L}[x] = \int_0^1 [x^\triangle(t)^2 - x^\triangle(t)^4] \triangle t, \quad x(0) = 0, \quad x(1) = 0 \tag{3.3}$$

on the time scale $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ (note that we need to add zero in order to have a closed set). Let us show that $\tilde{x}(t) = 0$, $0 \leq t \leq 1$ is a weak local minimum for (3.3). In the topology induced by $\|\cdot\|_1$ consider the open ball of radius 1 centred at $\tilde{x}$, i.e.,

$$B^1_1(\tilde{x}) = \left\{ x \in C^1_{prd} : \|x - \tilde{x}\|_1 < 1 \right\}.$$

We use the notation B^k_r for the ball of radius r in norm $\|\cdot\|_k$, $k = 1,2$. For every $x \in B^1_1(\tilde{x})$ we have

$$|x^\triangle(t)| \leq 1, \quad \forall t \in [0,1]^\kappa,$$

hence $\mathscr{L}[x] \geq 0$. This proves that $\tilde{x}$ is a weak local minimum for (3.3) since $\mathscr{L}[\tilde{x}] = 0$. Now let us consider the function defined by

$$x_d(t) = \begin{cases} d & \text{if } t = \sigma(t_0) \\ 0 & \text{otherwise} \end{cases}, \quad t_0 \in (0,1) \cap \mathbb{T}, \quad \sigma(t_0) \neq 1, \quad d \in \mathbb{R} \setminus \{0\}.$$

Function x_d is admissible and $\|x_d\|_0 = \sup_{t \in [0,1]^\kappa} |x_d^\sigma(t)| = |d|$. Therefore, for every $\delta > 0$ there is a d such that

$$x_d \in B^0_\delta(\tilde{x}) = \left\{ x \in C^1_{prd} : \|x - \tilde{x}\|_0 < \delta \right\}.$$

We have

$$x_d^\triangle(t_0) = \frac{d}{\mu(t_0)},$$

$$x_d^\triangle(\sigma(t_0)) = \frac{-d}{\mu(\sigma(t_0))},$$

and $x_d^\triangle(t) = 0$ for all $t \neq t_0, \sigma(t_0)$. Hence, $|x_d^\triangle(t)|$, $0 \leq t \leq 1$ can take arbitrary large values since $\mu(t) = \frac{t^2}{1-t} \to 0$ as $t \to 0$. Note that for every $\delta > 0$ we can choose d and t_0 such that $x_d \in B_\delta^0(\tilde{x})$ and $\frac{d}{\mu(\sigma(t_0))} > 1$. Finally,

$$
\begin{aligned}
\mathscr{L}[x_d] &= \int_0^1 [x_d^\triangle(t)^2 - x_d^\triangle(t)^4] \triangle t \\
&= \mu(t_0)\left[\left(\frac{d}{\mu(t_0)}\right)^2 - \left(\frac{d}{\mu(t_0)}\right)^4\right] + \mu(\sigma(t_0))\left[\left(\frac{d}{\mu(\sigma(t_0))}\right)^2 - \left(\frac{d}{\mu(\sigma(t_0))}\right)^4\right] \\
&= \frac{d^2}{\mu(t_0)}\left[1 - \frac{d^2}{\mu^2(t_0)}\right] + \frac{d^2}{\mu(\sigma(t_0))}\left[1 - \frac{d^2}{\mu^2(\sigma(t_0))}\right] < 0.
\end{aligned}
$$

Therefore, the trajectory $\tilde{x}$ cannot be a strong minimum for (3.3).

From now on we assume that $f : [t_0, t_1]^\kappa \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has partial continuous derivatives f_x and f_v, respectively with respect to the second and third variables, for all $t \in [t_0, t_1]^\kappa$, and $f(\cdot, x, v)$, $f_x(\cdot, x, v)$, and $f_v(\cdot, x, v)$ are continuous.

Let $E : [t_0, t_1]^\kappa \times \mathbb{R}^3 \to \mathbb{R}$ be the function with the values

$$E(t, x, r, q) = f(t, x, q) - f(t, x, r) - (q - r)f_r(t, x, r).$$

This function, called the *Weierstrass excess function*, is utilized in the following theorem:

Theorem 3.3 (Weierstrass necessary optimality condition on time scales). *Let $\mathbb{T}$ be a time scale, $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$. Assume that the function $f(t, x, r)$ in problem (3.1)–(3.2) satisfies the following condition:*

$$\mu(t)f(t, x, \gamma r_1 + (1 - \gamma)r_2) \leq \mu(t)\gamma f(t, x, r_1) + \mu(t)(1 - \gamma)f(t, x, r_2) \tag{3.4}$$

for each $(t, x) \in [t_0, t_1]^\kappa \times \mathbb{R}$, all $r_1, r_2 \in \mathbb{R}$ and $\gamma \in [0, 1]$. Let $\bar{x}$ be a piecewise continuous function. If $\bar{x}$ is a strong local minimum for (3.1)–(3.2), then

$$E[t, \bar{x}^\sigma(t), \bar{x}^\triangle(t), q] \geq 0 \tag{3.5}$$

for all $t \in [t_0, t_1]^\kappa$ and all $q \in \mathbb{R}$, where we replace $\bar{x}^\triangle(t)$ by $\bar{x}^\triangle(t-)$ and $\bar{x}^\triangle(t+)$ at finitely many points t where $\bar{x}^\triangle(t)$ does not exist.

Proof. Assume that $\bar{x}$ is a strong local minimum for (3.1)–(3.2). We consider two cases. First, suppose that $a \in [t_0, t_1]^\kappa$ is a right-scattered point. If $\bar{x}$ is a strong minimizer for problem (3.1)–(3.2), then the restriction of $\bar{x}$ to $[a, \sigma(a)] \cap \mathbb{T}$ is a strong minimizer for the problem (see [22])

$$\int_a^{\sigma(a)} f(t, x^\sigma(t), x^\triangle(t)) \triangle t \longrightarrow \min$$

$$x(a) = \bar{x}(a), \ x(\sigma(a)) = \bar{x}(\sigma(a)).$$

We define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(q) = \int_a^{\sigma(a)} f(t, \bar{x}^\sigma(t), q) \triangle t$. Hence, $h(q) = \mu(a)f(a, \bar{x}^\sigma(a), q)$. By assumption (3.4), we have immediately that

$$h(q) - h(\bar{x}^\triangle(a)) - (q - \bar{x}^\triangle(a))h'(\bar{x}^\triangle(a)) \geq 0.$$

This gives

$$E[a, \bar{x}^\sigma(a), \bar{x}^\triangle(a), q] \geq 0.$$

Second, we suppose that $a \in [t_0, t_1]^\kappa$, $a < t_1$, is a right-dense point and $[a, b] \cap \mathbb{T}$ is an interval between two successive points where $\bar{x}^\triangle(t)$ does not exist. Then, there exists a sequence $\{\varepsilon_k : k \in \mathbb{N}\} \subset [t_0, t_1]$ with $\lim_{k \to \infty} \varepsilon_k = a$. Let τ be any number such that $\sigma(\tau) \in [a, b)$ and $q \in \mathbb{R}$. We define the function $x : [t_0, t_1] \cap \mathbb{T} \to \mathbb{R}$ as follows:

$$x(t) = \begin{cases} \bar{x}(t) & \text{if } t \in [t_0, a] \cup [b, t_1], \\ X(t) & \text{if } t \in [a, \tau], \\ \phi(t, \tau) & \text{if } t \in [\tau, b], \end{cases}$$

where

$$X(t) = \bar{x}(a) + q(t - a), \quad q \in \mathbb{R},$$

$$\phi(t, \tau) = \bar{x}(t) + \frac{X(\tau) - \bar{x}(\tau)}{b - \tau}(b - t).$$

Clearly, given $\delta > 0$, for any q one can choose τ such that $\|x - \bar{x}\|_0 < \delta$. Let us now consider the function K defined for all $\tau \in [a, b) \cap \mathbb{T}$ such that $\sigma(\tau) \in [a, b) \cap \mathbb{T}$ with the values $K(\tau) = \mathscr{L}[x] - \mathscr{L}[\bar{x}]$. Since $\mathscr{L}[x] \geq \mathscr{L}[\bar{x}]$, by hypothesis, $K(\tau) \geq 0$ and $K(a) = 0$, it follows by Theorem 1.12 in [11] that $K^\triangle(a) \geq 0$. By the definition of x, we have

$$K(\tau) = \int_a^\tau \{f[t, X^\sigma(t), X^\triangle(t)] - f[t, \bar{x}^\sigma(t), \bar{x}^\triangle(t)]\}\triangle t$$

$$+ \int_\tau^b \{f[t, \phi(\sigma(t), \tau), \phi^{\triangle_1}(t, \tau)] - f[t, \bar{x}^\sigma(t), \bar{x}^\triangle(t)]\}\triangle t$$

so that, by Theorem 5.37 in [7] and Theorem 7.1 in [8], we obtain

$$K^\triangle(\tau) = f[\tau, X^\sigma(\tau), X^\triangle(\tau)] - f[\tau, \phi(\sigma(\tau), \sigma(\tau)), \phi^{\triangle_t}(\tau, \sigma(\tau))] \tag{3.6}$$

$$+ \int_\tau^b \{f_x[t, \phi(\sigma(t), \tau), \phi^{\triangle_1}(t, \tau)]\phi^{\triangle_2}(\sigma(t), \tau) + f_r[t, \phi(\sigma(t), \tau), \phi^{\triangle_1}(t, \tau)]\phi^{\triangle_1 \triangle_2}(t, \tau)\}\triangle t. \tag{3.7}$$

Invoking the relation $\phi^{\triangle_1 \triangle_2} = \phi^{\triangle_2 \triangle_1}$ (see Theorem 6.1 in [8]), integration by parts gives

$$\int_\tau^b f_r \phi^{\triangle_2 \triangle_1}(t, \tau)\triangle t = f_r \phi^{\triangle_2}(t, \tau)|_\tau^b - \int_\tau^b f_r^{\triangle_1} \phi^{\triangle_2}(\sigma(t), \tau)\triangle t.$$

Thus, (3.7) becomes

$$\int_\tau^b [f_x - f_r^{\triangle_1}]\phi^{\triangle_2}(\sigma(t), \tau)\triangle t$$

$$+ f_r \phi^{\triangle_2}(t, \tau)|_\tau^b. \tag{3.8}$$

From the definition of $\phi(t, \tau)$ we have

$$\phi^{\triangle_2}(t, \tau) = \frac{(X^\triangle(\tau) - \bar{x}^\triangle(\tau))(b - \tau) + X(\tau) - \bar{x}(\tau)}{(b - \tau)(b - \sigma(\tau))}(b - t)$$

so that $\phi^{\triangle_2}(b, a) = 0$, $\phi^{\triangle_2}(a, a) = X^\triangle(a) - \bar{x}^\triangle(a)$. Also, $\phi(\sigma(t), a) = \bar{x}(\sigma(t))$, $\phi^{\triangle_1}(t, a) = \bar{x}^\triangle(t)$. Thus, letting $\tau = a$ in (3.8) we obtain

$$-f_r[a, \bar{x}(\sigma(a)), \bar{x}^\triangle(a)][X^\triangle(a) - \bar{x}^\triangle(a)].$$

Since $\bar{x}$ verifies the Euler–Lagrange equation (see [6]), we get

$$\int_{\tau}^{b} \{f_x[t,\bar{x}(\sigma(t)),\bar{x}^{\triangle}(t)] - f_r^{\triangle}[t,\bar{x}(\sigma(t)),\bar{x}^{\triangle}(t)]\} \phi^{\triangle_2}(\sigma(t),\tau)\triangle t = 0.$$

On account of the above, from (3.6)–(3.7) we have

$$K^{\triangle}(a) = f[a,X^{\sigma}(a),X^{\triangle}(a)] - f[a,\phi(\sigma(a),\sigma(a)),\phi^{\triangle_1}(a,\sigma(a))]$$
$$- f_r[a,\bar{x}(\sigma(a)),\bar{x}^{\triangle}(a)][X^{\triangle}(a) - \bar{x}^{\triangle}(a)].$$

However, $X^{\sigma}(a) = \bar{x}^{\sigma}(a)$, $X^{\triangle}(a) = q$, $\phi(\sigma(a),\sigma(a)) = \bar{x}^{\sigma}(a)$, $\phi^{\triangle_1}(a,\sigma(a)) = \bar{x}^{\triangle}(a)$. Therefore,

$$K^{\triangle}(a) = f[a,\bar{x}^{\sigma}(a),q] - f[a,\bar{x}^{\sigma}(a),\bar{x}^{\triangle}(a)] - f_r[a,\bar{x}^{\sigma}(a),\bar{x}^{\triangle}(a)][q - \bar{x}^{\triangle}(a)],$$

and from this

$$K^{\triangle}(a) = E[a,\bar{x}^{\sigma}(a),\bar{x}^{\triangle}(a),q] \geq 0.$$

To establish the condition (3.5) for all $t \in [t_0,t_1]^{\kappa}$, we consider the limit $t \to t_1$ from left when t_1 is left-dense, and the limit $t \to t_p$ from left and from right when $t_p \in T$. $\qquad\square$

Remark 3.4. For $\mathbb{T} = \mathbb{R}$ problem (3.1)–(3.2) coincides with the classical problem of the calculus of variations. Condition (3.4) is then trivially satisfied and Theorem 3.3 is known as the Weierstrass necessary condition.

Remark 3.5. Let $\mathbb{T}$ be a time scale with $\mu(t)$ depending on t and such that the time scale interval $[t_0,t_1]$ may be written as follows: $[t_0,t_1] = L \cup U$ with $\mu(t) \neq 0$ for all $t \in L$ and $\mu(t) = 0$ for all $t \in U$. An example of such time scale is the Cantor set [10]. Then, for $t \in U$ condition (3.4) is trivially satisfied, while for $t \in L$ (3.4) is nothing more than convexity of f with respect to r.

4. SPECIAL CASES

Let $\mathbb{T} = \mathbb{Z}$. If $\bar{x}$ is a local minimum of the problem

$$\text{minimize} \quad \mathscr{L}[x] = \sum_{t=t_0}^{t_1-1} f(t,x(t+1),\triangle x(t)),$$

$$x(t_0) = \alpha, \ x(t_1) = \beta, \quad \alpha,\beta \in \mathbb{R},$$

and the function $f(t,x,r)$ is convex with respect to $r \in \mathbb{R}$ for each $(t,x) \in [t_0,t_1-1] \times \mathbb{R}$, then $E[t,\bar{x}(t+1),\triangle\bar{x}(t),q] \geq 0$ for all $t \in [t_0,t_1-1]$ and all $q \in \mathbb{R}$.

Let now $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$. If $\bar{x}$ is a local minimum of the problem

$$\text{minimize} \quad \mathscr{L}[x] = \sum_{t \in [t_0,t_1)} (q-1)tf\left(t,x(qt),\frac{x(qt)-x(t)}{qt-t}\right),$$

$$x(t_0) = \alpha, \ x(t_1) = \beta, \quad \alpha,\beta \in \mathbb{R},$$

and the function $f(t,x,r)$ is convex with respect to $r \in \mathbb{R}$ for each $(t,x) \in [t_0,t_1) \times \mathbb{R}$, then

$$E\left[t,\bar{x}(qt),\frac{\bar{x}(qt)-\bar{x}(t)}{qt-t},p\right] \geq 0$$

for all $t \in [t_0,t_1)$ and all $p \in \mathbb{R}$.

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Variatsiooniarvutuse tugevad minimeerijad ajaskaaladel ja Weierstrassi tingimus

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On sisse toodud tugeva lokaalse minimeerija mõiste variatsiooniarvutuseks ajaskaaladel. Lihtsate näidetega on demonstreeritud, et ajaskaala korral ei tarvitse nõrk miinimum olla ühtlasi ka tugev miinimum. Weierstrassi tarvilik optimeerimistingimus on tõestatud ajaskaala korral, mis sisaldab ja ühtlasi võimaldab üldistada saadud tulemust vastavatele tingimustele nii pideva kui diskreetse aja jaoks.