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On the propagation of localized perturbations in media with microstructure

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Abstract. The propagation of solitary waves in dilatant granular materials is studied using the hierarchical Korteweg–de Vries type evolution equation. The model equation is solved numerically under localized initial conditions by the pseudospectral method. The behaviour of the solution is analysed over a wide range of material parameters (two dispersion parameters and one microstructure parameter). Five solution types are introduced. Special attention is paid to the solitonic character of solutions.

Key words: dilatant granular materials, solitons, wave hierarchies, Korteweg–de Vries type evolution equations.

1. INTRODUCTION

Many physical and technological applications deal with nonlinear wave propagation in continuous media with microstructure. Granular materials could be one example of such microstructured materials [1–3]. The flow behaviour of a granular material is usually considered to be similar to the fluid behaviour, except that its response depends on the distribution of the volume fraction in the reference placement. The introduction of the volume fraction of the grains as an independent kinematic variable, in order to describe the local deformations of the grains themselves, requires an additional balance equation for the microinertia [3]. In dynamics the most important scale factor is an averaged diameter of the grain, which must be related to the wavelength of the excitation (i.e. propagating wave). A physically consistent derivation of the governing mathematical model of dilatant granular materials is given by Giovine and Oliveri [1]. In one-dimensional setting the governing equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha_1\frac{\partial^3 u}{\partial x^3} + \beta\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha_2\frac{\partial^3 u}{\partial x^3}\right) = 0. \tag{1}$$

Here the variable u is bulk density, x is the space coordinate, t is time, α_1 and α_2 are macro- and microlevel dispersion parameters, respectively, and β is a parameter involving the ratio of the grain size to the wavelength. Equation (1) consists of two Korteweg–de Vries (KdV) operators: the first describes the motion in the macrostructure and the second (in brackets) – the motion in the microstructure. Equation (1) is clearly hierarchical in Whitham's sense – if the parameter β is small, then the influence of the microstructure can be neglected and the wave "feels" only the macrostructure [4]. If, however, the parameter β is large, then only the influence of the microstructure "is felt" by the wave. Due to that kind of hierarchy Eq. (1) could be called the hierarchical Korteweg–de Vries (HKdV) equation.

The present paper deals with numerical simulation of the propagation of localized initial perturbations. Special attention is paid to the solitonic character of solutions.

2. STATEMENT OF THE PROBLEM

In the present paper the propagation of solitary waves in dilatant granular materials is studied making use of the HKdV equation (1). For this reason the model Eq. (1) is integrated numerically under localized initial and periodic boundary conditions

$$\begin{aligned} u(x,0) &= A\,\mathrm{sech}^2\,\frac{x}{\delta}, \qquad & \delta &= \sqrt{\frac{12\alpha_1}{A}}, \\ u(x+32k\pi,t) &= u(x,t), \qquad & k &= \pm 1, \pm 2, \pm 3, \ldots, \end{aligned} \tag{2}$$

where A is the amplitude and δ the width of the initial pulse. It is clear that the latter is the analytical solution of the KdV equation, which corresponds to the first KdV operator in Eq. (1) [5].

The goals of the present paper are: (i) to find numerical solutions to the proposed problem (1), (2) over a wide range of material parameters (dispersion parameters α_1 and α_2 and microstructure parameter β); (ii) to characterize the time–space behaviour of solutions and to define solution types; (iii) to analyse the results over the 3-dimensional domain of material parameters α_1, α_2, and β.

3. NUMERICAL METHOD

For numerical integration of the HKdV equation the pseudospectral method (PsM) [6,7] is applied. In a nutshell, the idea of the PsM is to approximate space derivatives by a certain global method – reducing thereby a partial differential equation to an ordinary differential equation (ODE) – and to apply a certain ODE solver for integration with respect to the time variable. In the present paper space derivatives are found making use of the discrete Fourier transform (DFT),

$$U(\omega,t) = Fu = \sum_{j=0}^{n-1} u(j\Delta x, t)\exp\left(-\frac{2\pi i j\omega}{n}\right), \tag{3}$$

where n is the number of space-grid points, $\Delta x = 2\pi/n$ the space step, i an imaginary unit, $\omega = 0, \pm 1, \pm 2, \ldots, \pm(n/2-1), -n/2$, and F denotes the DFT.

The usual PsM algorithm, derived for $u_t = \Phi(u, u_x, u_{2x}, \ldots, u_{nx})$ type equations, needs to be modified due to the existence of the mixed partial derivative in the HKdV equation (1). At first we rewrite the HKdV equation in the form

$$(u+\beta u_{2x})_t + (u + 3\beta u_{2x})\,u_x + (\alpha_1 + \beta u)\,u_{3x} + \beta\alpha_2 u_{5x} = 0, \tag{4}$$

introduce a new variable v, and apply properties of the DFT

$$v = u + \beta u_{2x} = F^{-1}\left[\left(1-\beta\omega^2\right)F(u)\right]. \tag{5}$$

Here F^{-1} denotes the inverse Fourier transform. Now the variable u and its derivatives can be expressed in terms of v:

$$u = F^{-1}\left[\frac{F(v)}{1-\beta\omega^2}\right], \qquad \frac{\partial^n u}{\partial x^n} = F^{-1}\left[\frac{(i\omega)^n F(v)}{1-\beta\omega^2}\right]. \tag{6}$$

Substituting expression (5) into Eq. (4), one can derive an explicit equation with respect to the time derivative v_t:

$$v_t = -(u+3\beta u_{2x})u_x - (\alpha_1 + \beta u)\,u_{3x} - \alpha_2\beta u_{5x}. \tag{7}$$

In Eq. (7) the variable u and all its space derivatives could be expressed in terms of v according to expressions (5) and (6). Therefore Eq. (7) can be considered as an ODE with respect to the variable v and could be integrated numerically making use of standard ODE solvers. In the present study calculations are carried out using the SciPy package [8]: for the DFT the FFTW [9] library and for the ODE solver the F2PY [10] generated Python interface to the ODEPACK Fortran code [11] are used.

4. RESULTS AND DISCUSSION

Numerical integration is carried out for $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$, and $\beta = 111.11, 11.111, 1.111, 0.111, 0.0111$. The number of space-grid points $n = 1024$ and the length of the time interval $t_f = 100$. Based on the analysis of numerical experiments, one can introduce five solution types.

The first solution type is a *single KdV soliton*, i.e., just a single KdV soliton emerges over time. This solution appears in all cases, where both dispersion parameters α_1 and α_2 have equal values. Different values for the initial amplitude A or microstructure parameter β do not change this behaviour. As Eq. (1) consists of two KdV operators that are tight through the second derivative and as the initial condition is the analytical solution of the KdV equation, the result is quite predictable. A typical solution for parameters $\alpha_1 = \alpha_2$ is presented in Fig. 1 in the form of a time-slice plot. In order to comprehend the present and the next figures adequately, one has to remember that: (i) periodic boundary conditions are applied to numerical integration and (ii) all wave profiles are plotted over two space periods. The solution could be defined to be a soliton as the initial condition is the analytical solution of the KdV equation.

The second solution type is a *KdV soliton ensemble*. Here a train of KdV solitons emerges. The number of generated solitons depends on the values of the macrostructure dispersion parameter α_1 and the microstructure dispersion parameter α_2. If α_1 increases and α_2 is fixed, then the number of solitons in

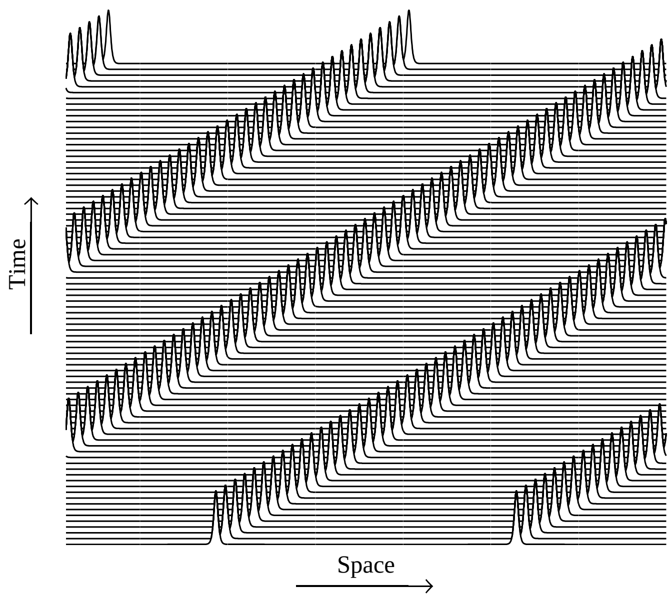

Fig. 1. The first solution type: a single KdV soliton. The time-slice plot over two space periods for $\alpha_1 = 0.07$, $\alpha_2 = 0.07$, $\beta = 11.111$, $n = 1024$, $t_f = 100$, $A = 4$.

the KdV ensemble increases. If α_1 is fixed and α_2 increases, then the number of solitons decreases. In Fig. 2 time evolution of an ensemble of two solitons is presented. The solution type is a soliton ensemble as the KdV solitons restore their shape and speed after interactions. The second solution type appears for $\alpha_2 < \alpha_1$ in the cases of $\beta = 111.11$ and $\beta = 11.111$.

The third solution type is a *KdV soliton ensemble with a weak tail*, i.e. a train of KdV solitons and a weak tail emerge (Fig. 3). The number of KdV solitons in the ensemble depends on the dispersion parameters α_1 and α_2 by the same rule as in the second type. The weakness of the tail is expressed through the fact that the tail does not influence the behaviour of the KdV ensemble essentially, i.e. here the behaviour of the KdV ensemble is similar to that of the second solution type (cf. Figs 2 and 3). The solution type is a soliton ensemble as the solitons in the train interact with each other and restore their shape and speed after the interactions. In this case the amplitude of the higher (if there are two solitons in the train) or highest (if the number of solitons in the train is higher than two) KdV soliton always increases compared to the initial amplitude A. The third solution type appears for $\alpha_2 < \alpha_1$ in the cases of $\beta = 111.11$ and $\beta = 11.111$, i.e., the third and the second solution types emerge in the same domain of parameters. However, for a fixed value of α_1 the third type is realized for higher values and the second type for lower values of the parameter α_2.

The fourth solution type is a *KdV soliton with a strong tail*, i.e. one KdV soliton and a strong tail emerge (Fig. 4). The behaviour of the solution is strongly

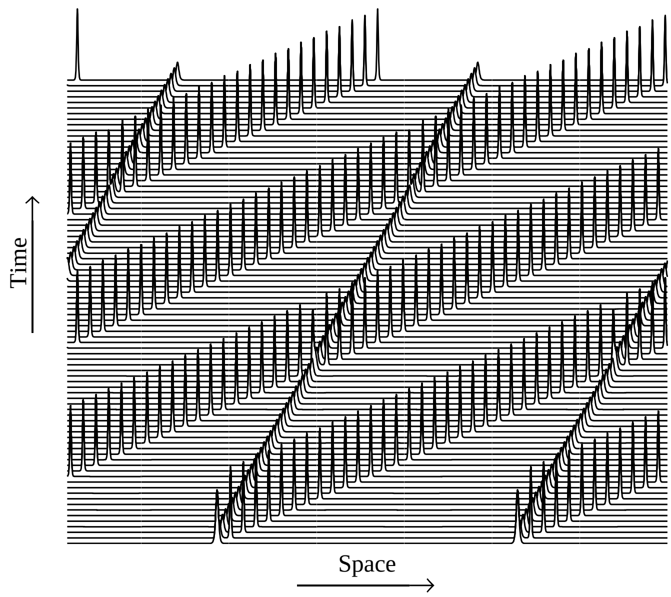

Fig. 2. The second solution type: a KdV soliton ensemble. The time-slice plot over two space periods for $\alpha_1 = 0.03$, $\alpha_2 = 0.01$, $\beta = 11.111$, $n = 1024$, $t_f = 100$, $A = 4$.

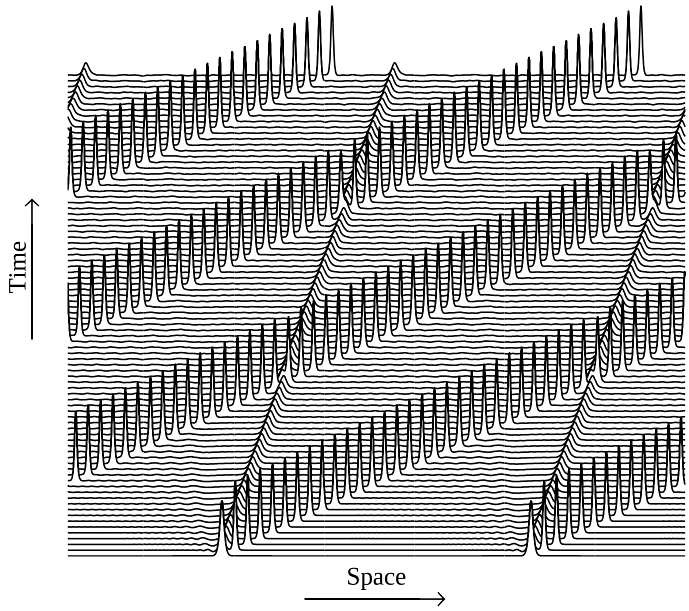

Fig. 3. The third solution type: a KdV soliton ensemble with a weak tail. The time-slice plot over two space periods for $\alpha_1 = 0.07$, $\alpha_2 = 0.03$, $\beta = 11.111$, $n = 1024$, $t_f = 100$, $A = 4$.

influenced by the tail – the amplitude of the KdV soliton is lower than the initial amplitude and the other KdV solitons are suppressed. Such a phenomenon is called selection in [12]. The fourth solution type emerges for $\alpha_2 > \alpha_1$ in the cases of $\beta = 111.11$ and $\beta = 11.111$, and for $\alpha_2 < \alpha_1$ in the case of $\beta = 0.0111$.

The fifth solution type is a *KdV soliton with a tail and wave package*, where one KdV soliton, a tail, and wave package(s) emerge simultaneously (Fig. 5). A similar situation is described by Christov and Velarde in [13]. The wave package is formed by several higher harmonics that are amplified. This phenomenon could be described similarly to a sum of two or more harmonic waves having nearly equal frequencies. The envelope of the package can propagate to the left or to the right and at a much higher speed than that of the KdV soliton or high-frequency waves that form the package. The solution is stable, i.e., all three components of the solution are conserved over long time intervals. As a rule, three different interactions take place: (i) KdV soliton – tail; (ii) KdV soliton – wave package; (iii) tail – wave package. Furthermore, in some cases two or more wave packages that propagate at different speeds emerge and therefore interactions between wave packages can take place. Like in the fourth solution type, a selection phenomenon [12] is observed and the amplitude of the propagating solitary wave is lower than the amplitude of the initial wave. The fifth solution type emerges for

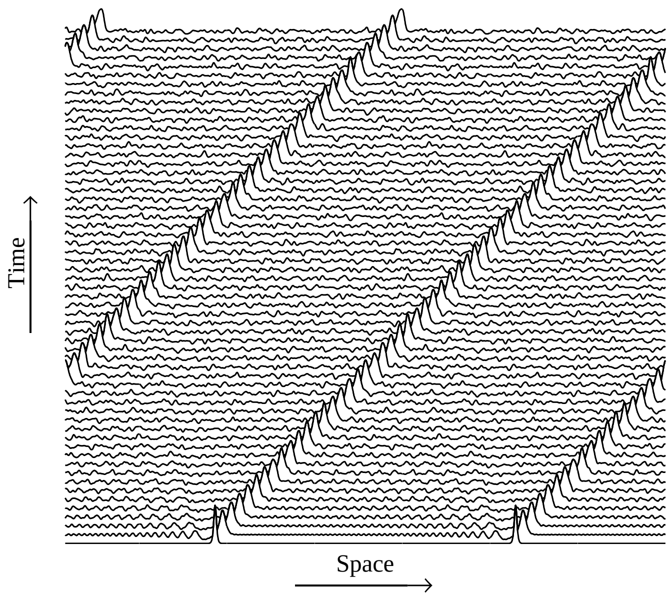

Fig. 4. The fourth solution type: a KdV soliton ensemble with a strong tail. The time-slice plot over two space periods for $\alpha_1 = 0.03$, $\alpha_2 = 0.09$, $\beta = 11.111$, $n = 1024$, $t_f = 100$, $A = 4$.

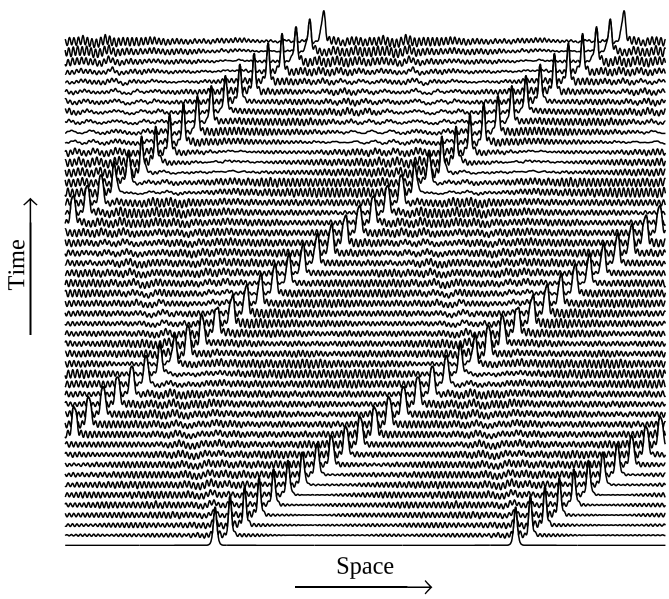

Fig. 5. The fifth solution type: a KdV soliton with a tail and wave package. The time-slice plot over two space periods for $\alpha_1 = 0.05$, $\alpha_2 = 0.07$, $\beta = 0.0111$, $n = 1024$, $t_f = 100$, $A = 4$.

$\alpha_2 \neq \alpha_1$ in the cases of $\beta = 1.111$ and $\beta = 0.111$, and for $\alpha_2 > \alpha_1$ in the case of $\beta = 0.0111$.

5. CONCLUSIONS

The main goal of the present paper was to study the propagation of solitary waves in dilatant granular materials. For this reason the HKdV equation (1) was integrated numerically under sech^2-type localized initial conditions and the time–space behaviour of solutions was analysed over a wide range of dispersion parameters α_1 and α_2, and microstructure parameter β.

Depending on the character of solutions, five solution types were defined: (i) a single KdV soliton, (ii) a KdV soliton ensemble, (iii) a KdV soliton ensemble with a weak tail, (iv) a KdV soliton with a strong tail, and (v) a KdV soliton with a tail and wave package. The first type is called a soliton, because the solitary wave (that propagates at a constant speed and shape) is the analytical soliton solution of the KdV equation that corresponds to both KdV operators in the HKdV equation (1). In the second and the third solution type a train of interacting solitary waves forms. The behaviour of the ensemble is very close to that of the KdV – solitary waves restore their speed and shape after interactions – and therefore they can be called solitons. In the second and the third solution type the amplitude of the highest soliton in the KdV ensemble is always higher than the amplitude of the initial pulse. A similar phenomenon can be observed for the KdV equation together with sech^2-type initial conditions. Therefore one can conclude that in the first three solution types the behaviour of the solution of the HKdV equation is very close to that of the KdV (remember that in case of the first solution type the initial pulse propagates with constant amplitude and speed).

In the fourth and the fifth solution type the name KdV soliton for the solitary wave is quite conditional, because no interactions between solitary waves take place. However, the single solitary wave interacts with the tail and wave packets and conserves its speed and shape through such interactions. In the fourth and fifth solution type the amplitude of the propagating KdV soliton is lower than the amplitude of the initial pulse.

The second and the third solution types are realized for higher values of β, i.e., when the influence on the microstructure is strong. The fifth solution type, vice versa, is realized when the influence on the microstructure is weak. The first and the fourth solution types can be realized in both cases. If the value of the parameter β is fixed, then the mutual ratio of the values of the parameters α_1 and α_2 determine the solution type and small changes in those values may cause a change in the solution type.

The most interesting phenomenon here is related to the fifth solution type, i.e., to the simultaneous emergence of the solitary wave, the tail, and the wave packet. Corresponding long-time simulations and detailed analysis are in progress and the results will be published in forthcoming papers.

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Lokaliseeritud häirituste levist mikrostruktuurses keskkonnas

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On uuritud lainelevi granuleeritud materjalides. Mudelvõrrandina on kasutatud Kortewegi-de Vriesi tüüpi hierarhilist evolutsioonivõrrandit, millele on leitud numbrilised lahendid lokaliseeritud algtingimusel ja perioodilistel rajatingimustel. Numbriliseks integreerimiseks on kasutatud pseudospektraalmeetodit. Lahendite käitumist on analüüsitud üle materjaliparameetrite (kaks dispersiooniparameetrit ja üks mikrostruktuuri parameeter) laia piirkonna. On defineeritud viis lahenditüüpi ja määratud vastavad piirkonnad materjaliparameetrite ruumis. Erilise tähelepanu all on lahendite solitoniline iseloom.