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Nonlinear waves in dissipative microstructured two-dimensional solids

Alessia Casasso[a], Franco Pastrone[a], and Alexander M. Samsonov[b]

[a] Dipartimento di Matematica, Università di Torino, Via C Alberto 10, Torino 10123, Italy; casasso@dm.unito.it, franco.pastrone@unito.it
[b] A. F. Ioffe Physico-Technical Institute, Russian Academy of Sciences, 194021 St Petersburg, Russia; samsonov@math.ioffe.rssi.ru

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Abstract. Plane Cosserat solids are introduced. It is shown that only one parameter is needed to describe intrinsic rotation. The field equations are simple enough to study the propagation of nonlinear waves and can be reduced to some particular form that admits different nonlinear waves, including solitons and cnoidal waves. We took into account both the matrix–grains and grain–grain interactions, nonlinearity, dispersion, and dissipation. Further examples can be provided of different kinds of analytical approaches, like asymptotical analysis, reduction to the Weierstrass equation, hierarchy of leading equations and waves. Solitary wave solutions and periodic bounded solutions are explicitly obtained.

Key words: microstructures, nonlinear waves, granular media, dissipation.

1. INTRODUCTION

The theory of continua with microstructures (see, e.g., [1,2]) can be used to model the behaviour of real materials such as granular materials, polycrystalline solids, ceramic composites, and materials with microdefects. One of the main features of the theory is the possibility of taking into account intrinsic space scales, namely, the size of grains, distance between microcracks, etc. Nonlinearity and dispersion cannot be avoided as phenomenological effects and mathematical features intrinsic to such phenomena. To describe granular materials, we must also introduce dissipation, which is mainly due to interaction between neighbouring grains.

The model of vector microstructures ([3–5]) seems to be useful for obtaining general field equations applicable to such materials. For simplicity, we restrict our attention to plane granular media. The natural model is the Cosserat theory, where the vector microstructure is described by a triad of orthonormal vectors. In field equations conservative stresses appear, i.e. stresses related to a generalized strain energy, and dissipative stresses, which are assumed to be linear in strain velocities.

Wave propagation can be studied by means of the well-known perturbation technique and slaving principle (see [6]). This approach allows us to reduce the set of general field equations to one equation in the ruling variable; hence we may estimate the possibility of propagation of solitary waves, depending on the balance of nonlinearity, dispersion, and dissipation. As a rule, dispersion is required for the existence of the bell-shaped solitary waves in an elastic microstructured medium, while dissipation is expected to be responsible for a saturation, which prevents unbounded growth of the bell-shaped solitary wave.

Our aim is to solve analytically the two-dimensional in space nonlinear wave propagation problem, using the simplest possible but informative description of internal dissipative features. Anaytical results are of considerable interest also as a tentative fault detection test for any numerical simulation in the 1+2D problem. The numerical solution in a 1+1D problem considered in [7] was based on the pseudospectral method supported by the analytical solution for the linear case to show how dissipative effects on various scales affect the harmonic wave.

We show how the influence of dissipation in the 1+2D problem in comparison with the simple case, where only nonlinearity and dispersion are taken into account, appears through a coefficient in the final solution, having a general form in terms of the Weierstrass elliptic function. It provides periodical and solitary wave solutions as appropriate limits. Moreover, it enters essentially the soliton solution to the 1+2D problem, and it seems feasible to claim that it can affect the amplitude evolution. Physical importance of the results obtained is in getting a detectable signal – the soliton, which is elusive in experiments without preliminary analytical solution (see [8,9]).

2. THE FIELD EQUATIONS

It is well known (see [3,4,10–12]) that the Cosserat solids can be described as nonlinear elastic solids C with a vectorial microstructure constrained to be a rigid triad $\mathbf{d}_\alpha = \mathbf{d}_\alpha(X^\beta, t)$, $\alpha, \beta = 1, 2, 3$, where X^β are material coordinates in a reference configuration C_*, such that $\mathbf{d}_\alpha \cdot \mathbf{d}_\beta = \delta_{\alpha\beta}$, $\forall\, t$.

In the plane case we have $\alpha, \beta = 1, 2$ and we can express the directors in terms of a rotation angle θ only:

$$\mathbf{d}_1 = \mathbf{d} = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \quad \mathbf{d}_2 = \nu = -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2, \tag{1}$$

where $\{\mathbf{e}_i\}$, $i = 1, 2$ is any orthonormal spatial basis, $0 \le \theta \le 2\pi$. We have

$$\dot{\mathbf{d}} = \dot{\theta}\nu, \quad \mathbf{d}_{,h} = \theta_{,h}\nu, \tag{2}$$

and writing $\mathbf{r} = x^h(X^h, t)\mathbf{e}_h$, we obtain for kinetic and strain energy densities

$$T = \frac{1}{2}\left(\rho\delta_{ij}\dot{x}^i\dot{x}^j + I\dot{\theta}^2\right) \quad \text{and} \quad W = W\left(x^h_{,k}, \theta, \theta_{,h}; X^h\right). \tag{3}$$

If we avoid body forces, the dissipative field equations yield ([13]):

$$\begin{cases} \rho\ddot{x}^h = \left(\dfrac{\partial W}{\partial x^h_{,i}} + \hat{\sigma}^h\right)_{,i}, \\[2ex] I\ddot{\theta} = \left(\dfrac{\partial W}{\partial \theta_{,i}} + \hat{\eta}^i\right)_{,i} - \left(\dfrac{\partial W}{\partial \theta} + \hat{\tau}\right), \end{cases} \tag{4}$$

where the terms $\hat{\sigma}^h$, $\hat{\eta}^i$, $\hat{\tau}$ represent the dissipation.

3. THE SIMPLER DISSIPATIVE CASE

We study the simplest model, where the dissipation is due to the rotation $\dot{\theta}$ only, through the term $\hat{\tau} = -F(\theta_x + \theta_y)_t$.

For the sake of simplicity, we shall use the notation: $X^1 = x$, $X^2 = y$, $x^1 = u$, $x^2 = v$. Hence we consider the vector $\mathbf{r} = \mathbf{r}(x, y, t) = u(x, y, t)\mathbf{e}_1 + v(x, y, t)\mathbf{e}_2$ for the macrostructure and, for the microstructure, the function $\theta = \theta(x, y, t)$ that represents the angle of rotation of the particle with respect to the fixed basis. In the following the subscripts x, y, t will denote differentiations.

The kinetic energy density reads

$$T = \frac{1}{2}\left[\rho\left(u_t^2 + v_t^2\right) + I\theta_t^2\right].$$

The strain energy density is chosen in the form

$$\begin{aligned} \mathcal{W} = &\frac{1}{2}\alpha\left(u_x^2 + v_x^2\right) + \frac{1}{2}\beta\left(u_y^2 + v_y^2\right) + \frac{1}{6}\gamma(u_x^3 + u_y^3 + v_x^3 + v_y^3) \\ &+ \frac{1}{2}\gamma(u_x^2 v_x + u_y^2 v_y + v_x^2 u_x + v_y^2 u_y) - A\theta\left(u_x + u_y + v_x + v_y\right) \\ &+ \frac{1}{2}B\theta^2 + \frac{1}{2}C\left(\theta_x^2 + \theta_y^2\right) + \frac{1}{3}D\left(\theta_x^3 + \theta_y^3\right). \end{aligned}$$

Since we are mainly interested in the estimation of θ, we introduce a new variable $U = u + v$. The field equations can be written as follows:

$$\begin{cases} \rho U_{tt} = \alpha U_{xx} + \beta U_{yy} - 2A(\theta_x + \theta_y) + \gamma\left[(U_x^2)_x + (U_y^2)_y\right], \\ I\theta_{tt} = C(\theta_{xx} + \theta_{yy}) + D\left[(\theta_x^2)_x + (\theta_y^2)_y\right] + A(U_x + U_y) - B\theta - F(\theta_x + \theta_y)_t. \end{cases} \tag{5}$$

Consider the dimensionless form of system (5), and apply the slaving principle (for all details see [13]). In addition to dimensionless quantity θ, for further analysis the dimensionless variables are introduced:

$$w = \frac{U}{w_0}, \quad X = \frac{x}{L}, \quad Y = \frac{y}{L}, \quad T = \frac{c_0^2}{L}t,$$

where c_0^2, w_0, L are physically meaningful constants (velocity, intensity, and wavelength of the initial excitation). We also need a scale for the microstructure l. Then two dimensionless parameters can be introduced: $\delta \sim \left(\frac{l}{L}\right)^2$ characterizing the relation between the microstructure and the wavelength and $\epsilon \sim \left(\frac{w_0}{L}\right)$ accounting for elastic strain, where δ is the relevant characteristic length.

Following [14], we suppose $I = \rho l^2 I^*$, $C = l^2 C^*$, $D = l^2 D^*$, $F = l^2 F^*$, where I^* is dimensionless and C^* and D^* have the dimension of the stress. We consider the expansion in terms of the characteristic length δ: $\theta = \theta_0 + \delta\theta_1 + ...$ and we impose the coefficients of powers of δ to be equal. Following the approximation for θ in terms of w and derivatives, we get

$$\begin{aligned}\theta \simeq \; & \frac{\epsilon A}{B}(w_X + w_Y) + \frac{\delta\epsilon A}{B^2}\Big[C^*(w_{XXX} + w_{XXY} + w_{XYY} + w_{YYY}) \\ & -\rho c_0^4 I^*(w_{XTT} + w_{YTT}) - c_0^4 F^*\Big[(w_X + w_Y)_{XT} + (w_X + w_Y)_{YT})\Big]\Big].\end{aligned}$$

Finally, we obtain the governing equation for w, i.e., for a nondimensional $U = u + v$:

$$\begin{aligned}w_{TT} = \; & \frac{1}{\rho c_0^4}(\alpha w_{XX} + \beta w_{YY}) - \frac{2A^2}{\rho c_0^4 B}(w_{XX} + 2w_{XY} + w_{YY}) \\ & - \frac{2\delta A^2 C^*}{\rho c_0^4 B^2}(w_{XXXX} + 2w_{XXXY} + 2w_{XXYY} + 2w_{XYYY} + w_{YYYY}) \\ & + \frac{2\delta A^2}{B^2}\left[I^*(w_{XX} + 2w_{XY} + w_{YY})_{TT} + c_0 F^*(w_{XXX} + 3w_{XXY}\right. \\ & \left. +3w_{XYY} + w_{YYY})_T\right] + \epsilon\gamma\left[\left(w_X^2\right)_X + \left(w_Y^2\right)_Y\right].\end{aligned}$$

Introducing the new variable $z = X + kY - cT$, we consider the 4th-order ODE for function $w(Z)$:

$$\tilde{\alpha}w'' + \tilde{\beta}(w'^2)' + \tilde{\gamma}w'''' = 0, \tag{6}$$

where we introduce the following coefficients $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$:

$$\tilde{\alpha} = \frac{B(\alpha + \beta k^2) - 2A^2(1+k)^2 - c^2\rho c_0^4 B}{\rho c_0^4 B}, \quad \tilde{\beta} = \epsilon\gamma(1 + k^3),$$

$$\tilde{\gamma} = \frac{2\delta A^2[I^* c^2 \rho c_0^4 (1+k)^2 - C^*(1 + 2k + 2k^2 + 2k^3 + k^4) - c\rho c_0^5 F^*(1+k)^3]}{\rho c_0^4 B^2}.$$

By setting $w' = y$, where $y = y(z)$, integrating twice Eq. (6) and rescaling the coefficients, we get the final equation:

$$(y')^2 + ay^2 + by^3 + dy + e = 0, \tag{7}$$

which is the Weierstrass equation, having *a general solution* in terms of $y = A\wp + B$, where $\wp = \wp(z + z_0; g_2; g_3)$. We define $y = A\wp + B, \ \to y' = A\wp'$; since $\wp'^2 = (4\wp^3 - g_2\wp - g_3)$, and have $(y')^2 = A^2(4\wp^3 - g_2\wp - g_3)$.

Replacing them into Eq. (7), we obtain

$$\begin{aligned} &A^2(4\wp^3 - g_2\wp - g_3) + a(A^2\wp^2 + B^2 + 2AB\wp) \\ &\quad + b(A^3\wp^3 + B^3 + 3A^2B\wp^2 + 3AB^2\wp) + d(A\wp + B) + e = 0. \end{aligned}$$

The coefficients A, B, g_2, g_3 are determined by making independently the coefficients of each order of $\wp$ and $\wp'$ equal to zero:

$$A = -\frac{4}{b}, \quad B = -\frac{a}{3b},$$

$$g_2 = -\frac{b}{4}\left[-\frac{a^2}{3b} + d\right], \quad g_3 = \frac{b^2}{16}\left(e - \frac{ad}{3b} + \frac{2a^3}{27b^2}\right).$$

Then

$$y = -\frac{4}{b}\wp(z + z_0;\ g_2;\ g_3) - \frac{a}{3b}, \tag{8}$$

where g_2 and g_3 are defined above. Eventually we have

$$y = -\frac{6\tilde{\gamma}}{\tilde{\beta}}\,\wp(z + z_0; g_2; g_3) + \frac{\tilde{\delta} + 6\tilde{\beta}\tilde{\gamma}}{6\tilde{\beta}\tilde{\gamma}}. \tag{9}$$

The last step of this approach consists in the evaluation of the integral:

$$w = \int y\,\mathrm{d}z = A\int \wp(z + z_0)\mathrm{d}z + Bz = -\frac{4}{b}\int \wp(z + z_0)\mathrm{d}z - \frac{a}{3b}z. \tag{10}$$

We consider, again, the expression (9), which is general, discontinuous, and semibounded from below or above. Solutions of this kind can be useful for tentative validation tests during numerical simulations. Of main physical interest are bounded continuous solutions; we obtain them as appropriate limits of (9) after some nontrivial algebra. Following [15], we introduce

(i) $2w$ and $2w'$ as primitive periods of $\wp$,

(ii) the discriminant $\Delta = g_2^3 - 27g_3^2$,

(iii) the roots $e_i = \wp(w_i)$, $i = 1, 2, 3$ of the equation $4e^3 - g_2e - g_3 = 0$, where $w_1 = w$, $w_2 = w + w'$, $w_3 = w'$.

Reduction of the doubly periodic Weierstrass function $\wp$ to a set of single periodic Jacobian elliptic functions is based on the following relationship between $\wp$ and cn, sn with modulus M:

$$\mathrm{sn}^2(z\sqrt{e_1-e_3}; M) = \frac{e_1-e_3}{\wp(z)-e_3}; \qquad \mathrm{cn}^2(z\sqrt{e_1-e_3}; M) = \frac{\wp(z)-e_1}{\wp(z)-e_3};$$

$$M = \sqrt{\frac{e_2-e_3}{e_1-e_3}}.$$

Moreover, (cf. [16]), if $e_1,\ e_2,\ e_3$ are any three numbers whose sum is zero, and if we write

$$y = e_3 + \frac{e_1-e_3}{\mathrm{sn}^2(z\sqrt{e_1-e_3}; M)},$$

we obtain the following relation between the Weierstrass function and the Jacobian elliptic function: $\wp(z; g_2; g_3) = e_3 + (e_1-e_3)\mathrm{ns}^2(z\sqrt{e_1-e_3)}|M)$. It is well known that the behaviour of $\wp$ depends on the sign and value of Δ, which allows us to extract two cases of main interest.

3.1. Two-dimensional solitary wave solution

In the case of $\Delta = 0$ one of the periods is infinite: $w = \infty$ or $w' = i\infty$ (the trivial case $w = -iw' = \infty$ will be excluded). The first case $w = \infty$ corresponds to $e_1 = e_2 \neq e_3$. Since $e_1 + e_2 + e_3 = 0$, introducing the condition $e_1 = e_2 \equiv E$, we have:

$$e_3 = -2E, \quad g_2 = 12E^2, \quad g_3 - 8E^3, \quad w' = i\pi/\sqrt{12E}.$$

Therefore the equation $y = -\dfrac{4}{b}\wp(z+z_0;\ g_2;\ g_3) - \dfrac{a}{3b}$ yields

$$\begin{aligned} y(z) &= -\frac{a}{3b} + e_3\left(-\frac{4}{b}\right) - \frac{4}{b}(e_1-e_3)\mathrm{ns}^2(z\sqrt{e_1-e_3}|M) \\ &= -\frac{a}{3b} + \frac{8E}{b} - \frac{12E}{b}\mathrm{sn}^{-2}(z\sqrt{3E}|M). \end{aligned}$$

In our case $M = 1$ and the limiting value $\mathrm{sn}(z, 1) = \tanh z$. Then we obtain *the exact 1+2D solitary wave solution* $y(z)$ to the original problem (9):

$$y(z) = -\frac{a}{3b} + \frac{8E}{b} - \frac{12E}{b}\tanh^{-2}(\sqrt{3E}(z+c_0)). \tag{11}$$

Integrating (10), where $y(z)$ is given by (11), we obtain finally

$$w = \left(-\frac{a}{3b} + \frac{8E}{b}\right)z + \frac{\sqrt{48E}}{b}\coth(\sqrt{3E}(z+c_0)).$$

The value $w' = \infty$ corresponds to $e_1 \neq e_2 = e_3$ and does not lead to bounded solutions (however, it very often occurs in numerical simulation).

In order to get the explicit expression for θ, we recall that

$$\begin{aligned}\theta =& \frac{\epsilon A}{B}(1+k)w' \\ &+ \frac{\delta\epsilon A}{B^2}\left\{\left[C^*(1+k+k^2+k^3) - \rho c_0^4 I^* c^2(1+k) - c_0 c F^*(1+k)^2\right] w'''\right\}.\end{aligned}$$

For simplicity we set

$$c_1 = \frac{\epsilon A}{B}(1+k),$$

$$c_2 = \frac{\delta\epsilon A}{B^2}\left[C^*(1+k+k^2+k^3) - \rho c_0^4 I^* c^2(1+k) - c_0 c F^*(1+k)^2\right],$$

and obtain $\theta = c_1 W' + c_2 W'''$, namely:

$$\begin{aligned}\theta =& \; c_1\left[-\frac{a}{3b} + \frac{8E}{b} - \frac{18}{b}\coth^2(\sqrt{3E}z)\right] \\ & -\frac{48E^2 c_2}{b}\operatorname{csch}^2(\sqrt{3E}z)\left[\operatorname{csch}^2(\sqrt{3E}z) + \coth^2(\sqrt{3E}z)\right].\end{aligned}$$

3.2. Periodic bounded solutions in the case $\Delta > 0$

In this case there is a pair of primitive periods $2w$ and $2w'$ such that w is a real and w' a pure imaginary semiperiod of the $\wp$ function. Assuming $\Delta > 0$, all roots e_i, $i = 1, 2, 3$ are real and different, $e_1 > e_2 > e_3$, $e_1 > 0$, $e_3 < 0$, from $y = \sqrt{-4/b}\,\wp(z + z_0;\ g_2;\ g_3) - \frac{a}{3b}$ we obtain the bounded periodical *1+2D cnoidal wave solution*:

$$\begin{aligned}y(z) &= -\frac{a}{3b} + \frac{4}{b}\wp(z; g_2; g_3) \\ &= -\frac{a}{3b} - \frac{4}{b}e_2 + (e_3 - e_2)\left(-\frac{4}{b}\right)\operatorname{cn}^2\left(\sqrt{e_1 - e_3}\,z|M\right).\end{aligned}$$

No bounded solutions $p(z)$ were found from (8) when $\Delta < 0$.

4. CONCLUSIONS

We aimed to show how the influence of dissipation in the nonlinear *two-dimensional problem* may be encapsulated in a compact form of the coefficient in the final ODE. From that equation we found a new general solution in terms of the Weierstrass elliptic function. It provides many periodical and solitary wave solutions as appropriate limits. The approach used does not require any reduction to evolutionary equation like the KP equation, which is widely used but restrictive in both derivation and solution of a two-dimensional nonlinear wave problem. Moreover, our approach yields an essentially new soliton solution to the *1+2D problem*. Thus it seems feasible to claim that the importance of our results in physics lies in the prediction of a detectable signal – the soliton, which is elusive in experiments without preliminary analytical solution.

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Mittelineaarsed lained dissipatiivsetes mikrostruktuursetes 2-dimensionaalsetes tahkistes

Alessia Casasso, Franco Pastrone ja Alexander M. Samsonov

On uuritud lainelevi granuleeritud keskkonnas. On kasutatud 2-dimensionaalset Cosserat' mudelit, kus on arvesse võetud graanulitevaheline vastasmõju ja keskkonna mittelineaarsed, dispersiivsed ning dissipatiivsed efektid. Saadud mudelvõrrand on kasutatav mittelineaarse lainelevi ülesannete lahendamiseks ja sellel on üldlahend, mis on esitatav Weierstrassi elliptiliste funktsioonide kaudu. Viimases sisaldub piirjuhtudel mitmeid perioodilisi, üksiklainelisi ja solitonilisi lahendeid.