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Wave equations in mechanics

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Abstract. The classical wave equation is a cornerstone in mathematical physics and mechanics. Its modifications are widely used in order to describe wave phenomena. In mechanics deformation waves are related to impact problems, acoustic waves are used in Nondestructive Evaluation, seismic waves may cause a lot of damage, etc. In this paper it is shown how the classical wave equation can be modified in order to model better the physics of processes. The examples cover microstructured and inhomogeneous materials together with linear and nonlinear models. Beside usual two-wave models, the evolution equations are described which govern the distortion of a single wave.

Key words: wave equation, modified wave equation, evolution equation.

1. INTRODUCTION

The cornerstones of classical mathematical physics are hyperbolic, parabolic and elliptic one-dimensional equations. Here we focus on one of them – the hyperbolic one which is called wave equation. From one side, the wave equation is one of 17 equations "that changed the world" [1], from the other side, it has an important role to play in mechanics. Indeed, mathematical description of wave phenomena is one of the fundamentals not only in mechanics but also in many other areas of physics. The history of the wave equation is related to such names as Jean d'Alembert, Leonhard Euler, Daniel Bernoulli, Luigi Lagrange and Joseph Fourier. The debate on proper solution of the wave equation between d'Alembert, Euler and Bernoulli during the 18th century has formulated the basics of the analysis and gave impetus to further studies [2]. "We live in a world of waves", said Stewart [1]. Sound waves, seismic waves, electromagnetic waves, etc. are known and studied intensively because they are around us, we can use them and sometimes we want to avoid them because they can be dangerous. In mechanics, we speak about deformation waves if we are interested in displacements and deformations, and about stress waves if we are interested in stresses. In terms of displacement u, the wave equation reads:

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where x, t are space and time coordinates and c_0 – the velocity of the wave (a constant). In three-dimensional setting in coordinates x, y, z the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{2}$$

Much is written about solving these equations in textbooks or monographs (see, for example, [[3]]). In a nutshell, the wave equation (1) describes the propagation of an excitation generated by initial or boundary condition with a constant speed c_0. There is no dissipation (which can not be realistic) and there are no constraints in time and space (which also cannot be realistic). Clearly, for most cases the wave equations must be modified to meet realistic conditions, but the essence of the model must be kept. The reason is simple: the wave equation emphasizes the Newton 2nd Law in continua and is the simplest version of balance of momentum involving kinetic and potential energies.

This paper gives a brief overview about modified wave equations, which are derived for bringing the models closer to reality. Without any doubt, such models are extremely important in engineering and acoustics for the analysis of dynamical phenomena like vibrations, impact processes, non-destructive testing, etc. Section 2 gives a brief overview on the physics of waves. Next sections are devoted to the description of mathematical models. In Section 3 the modifications of a classical wave equation are described, in Section 4 – the corresponding evolution equation is presented. Section 5 gives a brief summary of the importance of presented models.

2. PRELIMINARIES ON WAVES

There is no simple definition of a wave because of its many facets. Nevertheless, the following two definitions give a more or less clear picture. Truesdell and Noll [[4]] have said: wave is a state moving into another state with a finite velocity. One can also say [[5]]: wave is a disturbance, which propagates from one point in a medium to other points without giving the medium as a whole any permanent displacement. Following these definitions, it is clear that a wave should overcome the resistance of a medium to deformation and the resistance to motion. Consequently, waves can occur in media in which energy can be stored in both kinetic and potential form.

If in the simplest one-dimensional case we calculate kinetic energy $\mathcal{K}$ and potential energy W from

$$\mathcal{K} = \frac{1}{2}\rho u_t^2, \quad W = \frac{1}{2}(\lambda + 2\mu)u_x^2, \tag{3}$$

where ρ is the density; λ, μ are Lamé parameters and indices here and further denote differentiation with respect to variables x, t, then the wave equation (1) can be derived from Euler-Lagrange equations:

$$\rho u_{tt} = (\lambda + 2\mu)u_{xx}. \tag{4}$$

Here the left-hand side stems from the given kinetic energy resulting in acceleration and the right-hand side – from the given potential energy resulting in a force. It is easily seen that the velocity of the wave c_0 obeys the condition $c_0^2 = (\lambda + 2\mu)/\rho$.

The theory of deformation waves in solids was developed during the 19th century by Cauchy, Poisson, Lamé a.o. More recently, overviews on wave motion in solids were presented by Kolsky [6], Achenbach [7], Bland [8], Graff [9] a.o. (see also references to classical works therein). Note also an informative table on historical milestones in research into wave motion given by Graff [9].

In what follows we are interested in longitudinal waves, i.e. the particle motion is along the direction of propagation. The menagerie of waves includes also transverse waves, surface waves a.o. A brief summary of the types of waves is given by Engelbrecht [5].

The classical wave equation (1) has a closed solution (see [9]) for given initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{5}$$

Indeed, after introducing new variables

$$\xi = x + c_0 t, \quad \eta = x - c_0 t, \tag{6}$$

equations (1) yield

$$u_{\xi\eta} = 0, \tag{7}$$

which is solvable by direct integration. The final solution is named after d'Alembert:

$$u(x,t) = \frac{1}{2}\big(\varphi(x + c_0 t) + \varphi(x - c_0 t)\big) + \frac{1}{2c_0}\int\limits_{x-c_0 t}^{x+c_0 t} \psi(\alpha)d\alpha. \tag{8}$$

This solution shows explicitly waves propagating in two directions – to the left and to the right, which gives an idea to further factorization of the wave equation (see Section 4). Such an approach means replacing the wave equation (two-wave equation) by the corresponding one-wave equations, which are called evolution equations. For the classical wave equation such a replacement gives no advantage but for modified wave equations the evolution equations are widely used. The asymptotic techniques for constructing the evolution equations are well elaborated, especially for nonlinear waves [10,11].

3. MODIFIED WAVE EQUATIONS

The classical wave equation (1) (or as derived for solid mechanics – Eq. (4)) is certainly a simplification. It is linear because the disturbances are assumed to be small. Next, the medium is assumed to the homogeneous that is again a simplified assumption. In what follows, the description of modified wave equations follows. In order to be more definite, we limit ourselves mostly with elastic models, leaving dissipative effects aside. This means that the models are conservative and the attention is to nonlinear and dispersive effects. However, the modifications involving relaxation effects consider also dissipation. More on dissipative models can be found in [11].

First, we start with nonlinear models. The description of possible nonlinear effects in solids is given by Engelbrecht [5]. By taking geometrical (nonlinearity of the deformation tensor) and physical (nonlinearity of the stress–strain relation), the wave equation yields

$$\rho u_{tt} = (\lambda + 2\mu)\big[1 + 3(1 + m_0)u_x\big]u_{xx}, \tag{9}$$

where following the Murnaghan model

$$m_0 = 2(\nu_1 + \nu_2 + \nu_3)\big/(\lambda + 2\mu). \tag{10}$$

Here ν_1, ν_2, ν_3 are Murnaghan constants (higher-order elastic constants) corresponding to cubic terms in the potential energy W. In (9) the term $1 + m_0$ shows the influence of geometrical and physical nonlinearities (1 vs m_0). While for metals $|m_0| \sim 10$ [12] then the influence of physical nonlinearity is decisive for wave propagation. According to (1), the wave speed c is calculated from:

$$c^2 = c_0^2\big[1 + 3(1 + m_0)u_x\big], \tag{11}$$

which means that the speed depends upon deformation u_x. Consequently, under proper smooth initial/boundary conditions shock waves can emerge in the course of propagation [13]. Even if an initial excitation is a harmonic wave with a single frequency, in the course of propagation higher harmonics will be generated resulting in a shock wave which in mathematical terms is a singularity. Definitely, superposition is not possible in nonlinear systems.

Second, the solids at a smaller scale are heterogeneous because of their microstructure. There exist several theories of microstructured continua [14,15] and the corresponding mathematical models are characterized by the appearance of higher-order derivatives. The modified wave equation, for example, is presented and analysed by Engelbrecht et al. [16] and Berezovski et al. [17]. It reads:

$$u_{tt} = (c_0^2 - c_A^2)u_{xx} - p^2(u_{tt} - c_0^2 u_{xx})_{tt} + p^2 c_1^2(u_{tt} - c_0^2 u_{xx})_{xx}, \tag{12}$$

where c_A, c_1 are speeds characterizing elasticity of the microstructure and p is a coefficient characterizing microinertia.

After asymptotic simplification, Eq. (12) is transformed to

$$u_{tt} = (c_0^2 - c_A^2)u_{xx} + p^2 c_A^2 (u_{tt} - c_1^2 u_{xx})_{xx}. \tag{13}$$

Both Eq. (12) and Eq. (13) consist of the fourth-order terms characterizing dispersive effects. Their influence can be seen in wave profiles and phase and group speeds [17] as expected in dispersive systems.

Third, the nonlinear and dispersive effects, taken into account simultaneously, lead to the Boussinesq-type models, originally derived for water waves. In solids, such models are described by Christov et al. [18] and Engelbrecht et al. [19]. The Boussinesq paradigm grasps the following effects: (i) bi-directionality of waves; (ii) nonlinearity, which can be of any order; (iii) dispersion, modelled by space and time derivatives of the fourth order at least. If we now unite the models (9) and (13) derived for microstructured solids then the result yields (for details, see Engelbrecht et al. [19]):

$$u_{tt} = (c_0^2 - c_A^2)u_{xx} + \frac{1}{2}\mu(u_x^2) + p^2 c_A^2 (u_{tt} - c_1^2 u_{xx})_{xx} + \frac{1}{2}\delta^{3/2}\kappa(u_{xx}^2)_{xx}, \tag{14}$$

where μ and κ are nonlinear parameters and $\delta = l_0^2/L_0^2$ a small parameter, which determines the ratio of characteristic length l_0 of the microstructure and wavelength L_0 of the excitation.

In Eq. (14) both nonlinearities – that of the macrostructure and microstructure are taken into account (terms with coefficients μ and κ, respectively).

Fourth, the inhomogeneity of the material leads to a wave equation with space-dependent coefficients. Such a model is derived by Ravasoo [20] for solving the material characterization in nondestructive evaluation. In this case, all the coefficients of (9) are functions of x: $\rho(x), \lambda(x), \mu(x), \kappa_i(x), i = 1, 2, 3$. Then the final wave equation reads:

$$\rho(x)u_{tt} = \big[\lambda(x) + 2\mu(x)\big]\big(1 + k_1(x)\mu_x\big)u_{xx} + k_2(x)u_x + k_3(x)(u_x)^2. \tag{15}$$

Denoting

$$\alpha(x) = \lambda(x) + 2\mu(x); \quad \beta(x) = 2\big(\nu_1(x) + \nu_2(x) + \nu_3(x)\big), \tag{16}$$

the other coefficients are:

$$k_1 = 3\big[1 + \beta(x)/\alpha(x)\big], \tag{17}$$

$$k_2 = \alpha_x(x)/\alpha(x), \tag{18}$$

$$k_3 = \frac{3}{2}\big[\alpha_x(x) + \beta_x(x)\big]/\alpha(x). \tag{19}$$

Although Eq. (15) is pretty complicated, it is possible to solve it by perturbation techniques provided the space-dependence is weak [21].

Fifth, in order to model relaxation and/or hereditary effects, the constitutive equations (stress-strain relations) are taken in an integral form [22]. In this case the attention is to nonlocal effects [23] which are modelled by convolution integrals with certain kernel functions. Usually such models are used to describe dissipative effects but actually the summary effects are related also to wave speeds. That is why we present here also an integro-differential wave equation. In case of an exponential kernel function, the linear wave equation reads [11]:

$$\rho u_{tt} = (\lambda + 2\mu)\, u_{xx} + \epsilon_1 \,(\lambda + 2\mu) \left[\int_0^t (u_x)_\tau \exp\left(-\frac{t - \tau}{\tau_0} \right) d\tau \right]_x , \qquad (20)$$

or

$$\rho u_{tt} = (\lambda + 2\mu)(1 + \epsilon_1) u_{xx} - \epsilon_1 \,(\lambda + 2\mu)\,/\tau_0 \left[\int_0^t u_x \exp\left(-\frac{t - \tau}{\tau_0} \right) d\tau \right]_x . \quad (21)$$

Two new coefficients are here introduced: $\epsilon_1 > 0$ is a dimensionless constant and τ_0 – the relaxation time. From (20) and (21) two speeds can be determined: the equilibrium speed c_e and the instantaneous speed c_i:

$$c_e^2 = (\lambda + 2\mu)/\rho_0, \quad c_i^2 = (1 + \epsilon_1)(\lambda + 2\mu)/\rho_0. \qquad (22)$$

The slow processes propagate with speed c_e, the fast processes – with speed c_i [22]. This is an extremely important phenomenon in wave motion which demonstrates the possible dependence of wave characteristics on excitation properties. The possible nonlinear modifications of Eqs (20) and (21) are given by Engelbrecht [11]. The kernel of the convolution integral in Eqs (20) and (21) corresponds to the standard viscoelastic model [22], which models both dispersion and relaxation effects.

4. ONE-WAVE EQUATIONS

The wave equation (1) or its modifications describe two waves. The widely developed understanding in contemporary wave theory is related to the separation of a multi-wave process into separate waves. Then these single waves are governed by their own governing equations called evolution equations [10]. Although the most celebrated evolution equation named after Korteweg and de Vries was known much earlier, the systematic studies on derivation of such equations started in 70'ies last century.

The idea is as follows. The classical wave equation has solution in terms of variables $\xi = x + c_0 t, \eta = x - c_0 t$ and waves move without any distortion. Either ξ or η is then chosen as a basic independent variable also for modified wave equations (Section 3). A set of small parameters in then chosen which characterizes the strength of additional terms together with stretched independent variables and the perturbation method is applied [10,11].

For example, one can propose new variables

$$\xi = c_0 t - x, \quad \tau = \epsilon x, \tag{23}$$

where ϵ is a small parameter, and introduce series interpretation for dependent variables and coefficients. Note that new independent variables represent a moving frame: we move with a speed c_0 and the distortions of the wave profile are supposed to be slow. The sign convention in (23) says that the positive direction is from the wavefront backwards, i.e. into the waveprofile. Omitting the details [5,10,11], the classical wave equation (1) in new variables (23) is simply

$$v_\tau = 0, \tag{24}$$

where $v = u_t$ (or $v = -u_x c_0$). This equation reflects exactly the simplicity of the wave equation. The situation becomes much more complicated for modified wave equations. From nonlinear Eq. (9), the following evolution equation can be derived [11]:

$$v_\tau + n_1 v v_\xi = 0, \tag{25}$$

$$n_1 = \frac{3}{2}(1 + m_0)/\epsilon c_0. \tag{26}$$

Equation (25) is the equation of simple waves [24], which like Eq. (9) leads to shock waves.

From the original Boussinesq equation the famous Korteweg–de Vries (KdV) equation follows. Leaving aside the details how original variables are transformed to variables used throughout this paper, the KdV equation is

$$v_\tau + n_2 v v_\xi + d v_{\xi\xi\xi} = 0, \tag{27}$$

where n_2 and d are constants. The same KdV equation was derived by Zabusky and Kruskal [25] for waves in a chain of particles.

The canonical form of this equation after Newell [26] is

$$q_\tau + 6 q q_\xi + q_{\xi\xi\xi} = 0. \tag{28}$$

From (14), however, the following KdV-type equation can be derived [27]

$$v_\tau + n_3 (v^2)_\xi + (1 - \gamma_1^2) v_{\xi\xi\xi} + n_4 (v_\xi^2)_{\xi\xi} = 0. \tag{29}$$

Here $\gamma_1^2 = c_1^2/c_0^2$, the terms with n_3 and n_4 emphasize the influence of macro-nonlinearity and micro-nonlinearity, respectively. What should be stressed here is the sign of the dispersive term: $(1 - \gamma_1^2)v_{\xi\xi\xi}$. While in (14) the effects of microinertia and elasticity of the microstructure are controlled by different terms then in (29) there is only one term and the full description of dispersion is lost. Indeed, depending on either $\gamma_1^2 > 1$ or $\gamma_1^2 < 1$, the dispersion curve is concave or convex, respectively. However, the one-wave equation (29) is of the KdV-type and therefore demonstrates explicity the balance of dispersive and nonlinear terms needed for the existence of solitons. The counterpart to Eq. (28) is now

$$q_\tau + 3(q^2)_\xi + q_{\xi\xi\xi} + 3\epsilon(q_\xi^2)_{\xi\xi} = 0. \tag{30}$$

In case of the integro-differential models (20) and (21), the moving frame should be selected either in terms of the equilibrium velocity c_e or the instantaneous velocity c_i:

$$\xi_e = c_e t - x, \quad \xi_i = c_i t - x. \tag{31}$$

Then, for example from Eq. (20), the following evolution equation can be derived [5]:

$$v_\tau - \Xi_e \left[\int_0^{\xi_e} v_z \exp\left(-\frac{\xi_e - z}{Z_e}\right) dz \right]_{\xi_e} = 0. \tag{32}$$

Its nonlinear counterpart reads

$$v_\tau + \text{sign}(1 + m_0) v v_{\xi_e} - \Xi_e \left[\int_0^{\xi_e} v_z \exp\left(-\frac{\xi_e - z}{Z_e}\right) dz \right]_{\xi_e} = 0. \tag{33}$$

Here Ξ_e and Z_e are dimensionless parameters reflecting the properties of the medium and govern the dispersion of waves. Both Eqs (32) and (33) are special cases for hyperbolic waves analysed by Whitham [24]. In case of model (33), various simplifications are possible for large or small parameters Ξ_e and Z_e, which can be used to model low frequency ($Z_e \ll 1$) or high-frequency ($Z_e \gg 1$) processes [11].

5. FINAL REMARKS

Even the brief overview on possibilities to enlarge the classical wave equation in order to come closer to reality demonstrates the fundamental importance of the ideas embedded into the wave equation. As a direct consequence of the 2nd Newton's Law it is based on the kinetic and potential energies of the system. The cases shown above focus on the theory of elasticity and involve microstructured and inhomogeneous materials together with linear and nonlinear models. An example of a integro-differential model shows how the dispersive effects are linked to the relaxation process. The wave equation (1) itself is a two-wave equation and one possible simplification is to separate a multiwave processes into single waves which brings in the evolution equations. These one-wave equations describe the distortion of a single wave along a properly chosen characteristics, determined by the velocity c_0. Evolution equations form nowadays an important chapter not only in mechanics but also in many other areas of physics, especially when nonlinearities are included [28].

Theoretical modelling should always be verified by experiments. The wave equations and the corresponding evolution equations have stood all the verification. Here we note just one simple example related to the dynamics of the string [29].

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REFERENCES

1. Stewart, I. *17 Equations that Changed the World*. Profile, London, 2012.
2. Wheeler, G. F. and Crummelt, W. P. The vibrating string contraversy. *Am. J. Physics*, 1987, **55**, 1, 33–37.
3. Jost, J. *Partial Differential Equations*. Springer, New York et al., 2002.
4. Truesdell, C. and Noll, W. The non-linear field theories of mechanics. In *Enc. of Physics* (Flügge, S., ed.), III/3, Springer, Berlin et al., 1965.
5. Engelbrecht, J. *Nonlinear Wave Dynamics. Complexity and Simplicity*. Kluwer, Dordrecht, 1997.
6. Kolsky, H. *Stress Waves in Solids*. 2nd ed. Dover, New York, 1963.
7. Achenbach, J. D. *Wave Propagation in Elastic Solids*. North Holland, Amsterdam, 1973.
8. Bland, D. R. *Wave Theory and Applications*. Clarendon Press, Oxford, 1988.
9. Graff, K. F. *Wave Motion in Elastic Solids*. Dover, New York, 1991 (1st ed. 1975).
10. Taniuti, T. and Nishihara, K. *Nonlinear Waves*. Pitman, London et al., 1983 (in Japanese 1977).
11. Engelbrecht, J. *Nonlinear Wave Processes of Deformation in Solids*. Pitman, London, 1983.
12. Cantrell, J. H. Acoustic nonlinearity parameters and higher-order elastic constants for crystals. *Proc. IOA*, 1989, **11**, Part 5, 445–452.
13. Naugolnykh, K. and Ostrovsky, L. *Nonlinear Wave Processes in Acoustics*. Cambridge University Press, Cambridge, 1998.
14. Mindlin, R. D. Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.*, 1964, **16**, 51–78.
15. Eringen, A. C. *Microcontinuum Field Theories I: Foundations and Solids*. Springer, New York et al., 1999.
16. Engelbrecht, J., Berezovski, A., Pastrone, F. and Braun, M. Waves in microstructured materials and dispersion. *Phil. Mag.*, 2005, **85**, 33–35, 4127–4141.

17. Berezovski, A., Engelbrecht, J., Salupere, A., Tamm, K., Peets, T. and Berezovski, M. Dispersive waves in microstructured solids. *Int. J. Solids Structures*, 2013, **50**, 1981–1990.
18. Christov, C., Maugin, G. and Porubov, A. V. On Boussinesq's paradigm in nonlinear wave propagation. *C.R. Mécanique*, 2007, **335**, 521–535.
19. Engelbrecht, J., Salupere, A. and Tamm, K. Waves in microstructured solids and the Boussinesq's paradigm. *Wave Motion*, 2011, **48**, 717–726.
20. Ravasoo, A. Nonlinear waves in characterization of inhomogeneous elastic material. *Mech. Mater.*, 1999, **31**, 205–213.
21. Ravasoo, A. Interaction of bursts as a detector of material inhomogeneity. *Acta Acustica united with Acustica*, 2012, **98**, 864–869.
22. Chen, P. J. Growth and decay of waves in solids. In *Enc. of Physics* (Flügge, S., ed.), VI/3, Springer, Berlin, 1973.
23. Engelbrecht, J. and Braun, M. Nonlinear waves in nonlocal media. *Appl Mech. Rev.*, 1998, **51**, 475–488.
24. Whitham, G. *Linear and Nonlinear Waves*. Wiley-Interscience, New York, 1974.
25. Zabusky, N. and Kruskal, M. D. Interactions of 'solitons' in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 1965, **15**, 240–243.
26. Newell, A. C. *Solitons in Mathematics and Physics*. SIAM, Philadelphia, 1985.
27. Randrüüt, M. and Braun, M. On one-dimensional solitary waves in microstructured solids. *Wave Motion*, 2010, **47**, 4, 217–230.
28. Ablowitz, M. J. *Nonlinear Dispersive Waves*. Cambridge University Press, Cambridge, 2011.
29. Amstead, D. C. and Karls, M. A. Does the wave equation really work? *Primus*, 2006, XVI, No. 2, 162–176.

Lainevõrrandid mehaanikas

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Klassikaline lainevõrrand on üks matemaatilise füüsika põhivõrranditest. Selle võrrandi lihtsusest ja elegantsusest hoolimata on paljude mehaanikaprobleemide analüüsiks vaja kasutada lainevõrrandi modifikatsioone, et paremini kirjeldada füüsikalisi efekte laineleviprotsessides. Vajadus on eriti ilmne lainelevi modelleerimisel heterogeensetes materjalides, kus materjali sisestruktuuril on oluline osa. Ühemõõtmeline lainevõrrand ise ja selle modifikatsioonid kirjeldavad kaht lainet, kuid tihti on kasutusel ka nendest tuletatud nn ühe laine evolutsioonivõrrandid. Artiklis on toodud näiteid modifitseeritud lainevõrranditest ja nendele vastavatest evolutsioonivõrranditest. On rõhutatud vajadust arvestada dispersiooni ja materjali mittelineaarsusega.