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Stability of discrete-time systems via polytopes of reflection vector sets

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Abstract. The stability domain of discrete-time systems is investigated via reflection coefficients of characteristic polynomials of the system. Stable polytopes in the coefficients space of characteristic polynomials are defined starting from the sufficient stability condition in the polynomial reflection coefficients space using different reflection vector sets. The volumes of these stable polytopes are calculated via the triangulation method.

Key words: discrete-time systems, stability, polynomials.

1. INTRODUCTION

Convex approximation of the stability region in the polynomial coefficients space is a useful tool for many parametric robust control tasks [1,2]. Much research work has been done to approximate the Schur stability domain by boxes [3,4], ellipsoids [5,6], polytopes [7–9] or other convex sets [10,11]. In [12] a linear Schur invariant transformation with a free parameter is introduced in the discrete polynomial coefficient space which give us a possibility to generalize polytopic stability conditions such as Cohn's condition [1], discrete Kharitonov's theorem [13] and reflection vector polytopes [9].

An attempt to approximate the stability region in the polynomial coefficient space via reflection vector polytopes was made in [9,14]. In this paper, a similar approach will be used, but the starting point is more general: instead of a fixed reflection vector polytope we are studying stable families of different reflection vector sets.

We investigate the geometry of stable discrete polynomials using their coefficients and reflection coefficients. The reflection coefficients are also known in the literature as Schur–Szegö parameters [15], partial correlation (PARCOR) coefficients [16], k-parameters [17] or FM-parameters [18,19]. They have been used efficiently in many applications in signal processing [17], system identification [16] and robust control [14,18,19]. The reflection coefficient method is numerically fast and stable to generate stable polynomials [19] and it gives a simple and efficient approach to construct fixed-structure stabilizing controllers [18].

The aim of the paper is to find less conservative inner approximations of the stability domain by polytopes, starting from different sets of reflection vectors, generated by simple stable polynomials. The volumes of these stable polytopes are calculated in order to compare the approximation quality.

The paper is organized as follows. First, stable reflection vector polytopes are defined and their volume is investigated. Second, stable polytopes of two reflection vector sets are defined, and in the last section the volumes of these stable polytopes are calculated.

2. STABLE REFLECTION VECTOR POLYTOPES

Let $a^n(z)$ be a monic polynomial of degree n with real coefficients $a_i \in \mathcal{R}$, $i = 0, ..., n-1$

$$a^n(z) = z^n + ... + a_1 z + a_0.$$

The reverse-order polynomial $a^{n*}(z)$ of $a^n(z)$ is defined by [15]

$$a^{n*}(z) = a_0 z^n + ... + a_{n-1} z + 1.$$

The reflection coefficients k_i , $i = 1, ..., n$ can be obtained from $a^n(z)$ by using backward Levinson's recursion [20]

$$za^{i-1}(z) = \frac{1}{1-k_i^2}[a^i(z) + k_i a^{i*}(z)], \tag{1}$$

where $k_i = -a_0^i$ (a_0^i denotes the last coefficient of an ith-degree polynomial $a^i(z)$). From Eq. (1) the forward recursion can be obtained

$$a^i(z) = za^{i-1}(z) - k_i a^{(i-1)*}(z). \tag{2}$$

In the following, the index of the degree will be omitted for the sake of readability, i.e., $a^n(z) = a(z)$.

The stability criterion via reflection coefficient is as follows [15]: a polynomial $a(z)$ has all its roots inside the unit disk if and only if $|k_i| < 1$, $i = 1, ..., n$.

The reflection vectors of a Schur stable monic polynomial $a(z)$ are defined as the end points of stable line segments $\mathcal{A}^i(\pm 1) = conv\{a | k_i = \pm 1\}$ [9]

$$v^i(\pm 1) = (a | k_i = \pm 1), i = 1, ..., n,$$

where $conv\{a|k_i = \pm 1\}$ denotes the convex hull, obtained by varying the reflection coefficient k_i between -1 and 1 while all the other reflection coefficients are fixed.

The following assertions hold:

1) every Schur polynomial $a(z)$ has $2n$ reflection vectors $v_i^+(a)$ and $v_i^-(a)$, $i = 1, ..., n$;
2) all the reflection vectors of a Schur polynomial $a(z)$ lie on the stability boundary $(k_i(v) = \pm 1, k_j(v) \in (-1, 1), j = 1, ..., n; j \neq i)$;
3) the line segments between reflection vectors $v_i^+(a)$ and $v_i^-(a)$ of a Schur polynomial $a(z)$ are stable.

The following Lemma defines a family of stable polynomials such that a polytope, generated by the reflection vectors of these polynomials, is stable [9].

Lemma. *Let the reflection coefficients of a polynomial $a(z)$ be $k_1 \in (-1, 1)$ and $k_2 = ... = k_n = 0$. Then the innerpoints of the polytope $\mathcal{P}(a)$, generated by the reflection vectors of the polynomial $a(z)$*

$$\mathcal{P}(a) = conv\{v_i^{\pm}(a), \quad i = 1, ..., n\}, \tag{3}$$

are Schur stable.

Indeed, if $k = \begin{bmatrix} k_1 & 0 & ... & 0 \end{bmatrix}$ for the generating polynomial $a(z)$ then the reflection vectors (RV) $v_i^{\pm}(a)$ in n-dimensional coefficient space are presented as the rows of the following matrix:

$$P(a) = \begin{bmatrix} v_1^-(a) \\ v_1^+(a) \\ v_2^-(a) \\ v_2^+(a) \\ v_3^-(a) \\ v_3^+(a) \\ ... \\ v_n^-(a) \\ v_n^+(a) \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & ... & ... & ... & ... & 0 \\ -1 & 0 & ... & ... & ... & ... & ... & 0 \\ -2k_1 & 1 & 0 & ... & ... & ... & ... & 0 \\ 0 & -1 & 0 & ... & ... & ... & ... & 0 \\ -k_1 & -k_1 & 1 & 0 & ... & ... & ... & 0 \\ -k_1 & k_1 & -1 & 0 & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ -k_1 & 0 & ... & ... & ... & 0 & -k_1 & 1 \\ -k_1 & 0 & ... & ... & ... & 0 & k_1 & -1 \end{bmatrix}. \tag{4}$$

Here RV for $i > 2$ can be grouped in pairs of the form $v_i^{\pm}(a) = [-k_1 \quad 0 \quad ... \quad 0 \quad \pm k_1 \quad \pm 1 \quad 0 \quad ... \quad 0]$. The RV created by positive $k_i = 1$ vector will be called a "*positive RV on level i*"

$$v_i^+(a) = \ [-k_1 \quad 0 \quad ... \quad 0 \quad k_1 \quad -1 \quad 0 \quad ... \quad 0],$$

while the RV formed by $k_i = -1$ will be called a "*negative RV on level i*"

$$v_i^-(a) = \ [-k_1 \quad 0 \quad ... \quad 0 \quad -k_1 \quad 1 \quad 0 \quad ... \quad 0].$$

Now let us investigate the volume of stable polytopes $\mathcal{P}(k)$, generated by the rows of the matrix $P(a)$ for $k_1 \in (-1, 1)$ and $n \geq 2$.

Theorem 1. *The volume of a reflection vector polytope* $\mathcal{P}(a) \in \mathcal{R}^n$ *of a polynomial* $a(z)$*, generated by reflection coefficients* $k = [k_1 \ 0 \ \dots \ 0]$ *, is fixed for* $k_1 \in (-1, 1)$ *by the dimension* n*:*

$$V(\mathcal{P}(a)) = \frac{2^n}{n!}. \tag{5}$$

Proof. Any polytope $\mathcal{P}$ in n-dimensional space can be decomposed into a number of simplexes with $(n+1)$ vertices v_i, $i = 1, ..., n$ in each of them and a single common vertex $v_0 \in \mathcal{P}$ for all the simplexes. The volume of a simplex S is [21]

$$V(S) = \frac{|det(v_1 - v_0; v_2 - v_0; ...; v_n - v_0)|}{n!}, \tag{6}$$

where $S = [v_0 \ \ v_1 \ \ \dots \ \ v_n]^T$ denotes a matrix of $n+1$ vertices (row vectors) $v_0, ..., v_n \in \mathcal{R}^n$ of a simplex.

The volume of a polytope is then

$$V(P) = \sum_{j=1}^{s} V(S_j),$$

where s is the number of simplexes in the polytope.

For an arbitrary RV polytope $\mathcal{P}(a)$ (4) the origin is an inner point of the polytope $\mathcal{P}(a)$, $0 \in \mathcal{P}(a)$. So we can choose origin as the common vertex for all the decoupling simplexes S_j, $v_0 = [0 \ \ 0 \ \ \dots \ \ 0] \in S_j$, $j = 1, ...s$. According to the definition of reflection vectors and the polytope decomposition conditions, all the other vertices $v_1, ..., v_n$ of a decoupling simplex can not be on the same level i (4).

As there are n levels in a RV polytope, then each simplex will include one vertex of every level of $\mathcal{P}(a)$, i.e. $v_1 = [v_1^-, v_1^+], v_2 = [v_2^-, v_2^+], ..., v_n = [v_n^-, v_n^+]$. Therefore, the total number of simplexes in the polytope is $s = 2^n$.

All the decoupling simplexes S_j, $j = 1, ...s$, have the structure of a lower triangular matrix

$$S_j = \begin{bmatrix} \pm 1 & 0 & \dots & \dots & \dots & 0 \\ -k_1 \pm k_1 & \mp 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -k_1 & \dots & \pm k_1 & \mp 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -k_1 & 0 & \dots & 0 & \pm k_1 & \mp 1 \end{bmatrix}$$

and so the volume of a simplex is according to Eq. (6)

$$V(S_j) = \frac{1}{n!}, \quad j = 1, ..., s.$$

As the total number of simplexes in a RV polytope is $s = 2^n$ then the volume of an arbitrary RV polytope (5) is

$$V(\mathcal{P}(a)) = \sum_{j=1}^{s} V(S_j) = \frac{2^n}{n!}.$$

It proves the Theorem. □

Example. Let us consider second order polynomials $n = 2$. Then the stability domain in the polynomial coefficients space $a \in \mathcal{R}^2$ is given by the triangle $(\mathbf{F}, \mathbf{G}, \mathbf{H})$ in Fig. 1. Let us choose according to Theorem 1 the reflection coefficients $k_1 = 0.2$ and $k_2 = 0$. Then the generating polynomial $a(z) = z^2 - 0.2z$ has 4 reflection vectors (points $\mathbf{C}, \mathbf{D}, \mathbf{F}, \mathbf{A}$ in Fig. 1, respectively)

$$\begin{array}{rcl} v_1^+(a) & = & [\quad -1 \qquad 0 \quad], \\ v_1^-(a) & = & [\quad 1 \qquad 0 \quad], \\ v_2^+(a) & = & [\quad 0 \qquad -1 \quad], \\ v_2^-(a) & = & [\quad -0.4 \qquad 1 \quad]. \end{array}$$

The RV polytope $\mathcal{P}(a)$ $(\mathbf{C}, \mathbf{A}, \mathbf{D}, \mathbf{F})$ can be decoupled into four simplexes $(\mathbf{C}, \mathbf{A}, \mathbf{0})$, $(\mathbf{A}, \mathbf{D}, \mathbf{0})$, $(\mathbf{D}, \mathbf{F}, \mathbf{0})$ and $(\mathbf{F}, \mathbf{C}, \mathbf{0})$. All of these four triangles have the volume

$$V(S_j) = \frac{1}{2!}, \qquad i = 1, ..., s,$$

and the volume of the RV polytope $(\mathbf{C}, \mathbf{A}, \mathbf{D}, \mathbf{F})$ is

$$V(\mathcal{P}(a)) = \sum_{i=1}^{4} V(S_j) = \frac{2^2}{2!} = 2.$$

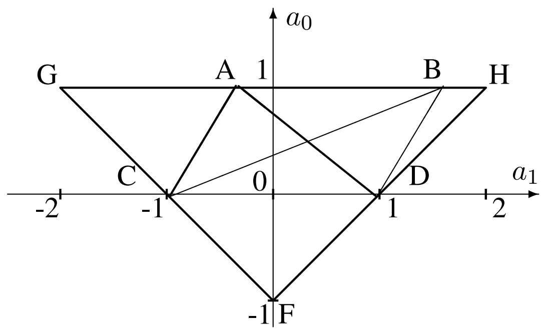

Fig. 1. Polytopes of reflection vectors.

To illustrate the effect of the choice of reflection coefficients of the generating polynomial $a(z)$ we have solved the same task with $k_1 = -0.8$. Then the RV polytope $\mathcal{P}(a)$ is the quadrangle $(\mathbf{C}, \mathbf{B}, \mathbf{D}, \mathbf{F})$ with decoupling triangles $(\mathbf{C}, \mathbf{B}, \mathbf{0})$, $(\mathbf{B}, \mathbf{D}, \mathbf{0})$, $(\mathbf{D}, \mathbf{F}, \mathbf{0})$, $(\mathbf{F}, \mathbf{C}, \mathbf{0})$ and the volume

$$V(\mathbf{CBDF}) = V(\mathbf{CADF}) = 2.$$

3. STABILITY OF THE POLYTOPE OF TWO REFLECTION VECTOR SETS

Let us consider now the reflection vector polytopes $\mathcal{P}(a)$ and $\mathcal{P}(\tilde{a})$ of two polynomials $a(z)$ and $\tilde{a}(z)$ with reflection coefficients $k = [\ k_1 \ \ 0 \ \ ... \ \ 0\ ]$ and $\tilde{k} = [\ \tilde{k}_1 \ \ 0 \ \ ... \ \ 0\ ]$, respectively. By Lemma, both of them are stable if $k_1 \in (-1, 1)$ and $\tilde{k}_1 \in (-1, 1)$. Then the union of reflection vector polytopes $\mathcal{P}(a) \bigcup \mathcal{P}(\tilde{a})$ is also stable. Unfortunately, the union of polytopes is not a convex set. That is why we are looking for a stable convex hull of the two sets of reflection vectors.

Let us denote by $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ the polytope generated by the reflection vectors $v_i^{\pm}(a)$ and $v_i^{\pm}(\tilde{a})$, $i = 1, ..., n$ of two polynomials $a(z)$ and $\tilde{a}(z)$

$$\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a}) = conv\{v_i^{\pm}(a), v_i^{\pm}(\tilde{a}), i = 1, ..., n\},$$

and call it *the polytope of two reflection vector sets*. In this section we study the stability of the polytopes $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ starting from a single stable reflection vector polytope $\mathcal{P}(a)$ with $k_1 \in (-1, 1)$, $\tilde{k}_1 = k_1$ and increasing $\tilde{k}_1 > k_1$ as much as possible.

The following theorem gives us a useful tool for generating stable polytopes of two RV sets with maximal volume.

Theorem 2. *If the polytope of reflection vector sets $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ is stable and $-1 < k_1 < \bar{k}_1 < \tilde{k}_1 < 1$, then the polytope of reflection vector sets $\mathcal{P}(a) \biguplus \mathcal{P}(\bar{a})$ is stable.*

Proof. By Lemma all the three of reflection vector polytopes $\mathcal{P}(a)$, $\mathcal{P}(\tilde{a})$ and $\mathcal{P}(\bar{a})$ are stable. In order to prove the Theorem via Edge Theorem [22] we have to prove that all the edges between all the vertices of the polytopes $\mathcal{P}(a)$ and $\mathcal{P}(\bar{a})$ are stable.

Let us consider now an edge $\mathcal{A}_{k,\bar{k}}^{ij}(\pm 1) = conv\{v_i^{\pm}(a), v_j^{\pm}(\bar{a})\}$ between arbitrary vertices $v_i^{\pm}(a)$ and $v_j^{\pm}(\bar{a})$, $i, j = 1, ..., n$ of reflection vector polytopes $\mathcal{P}(a)$ and $\mathcal{P}(\bar{a})$, respectively.

It is easy to see that the line segment $\mathcal{A}_{k,\bar{k}}^{ij}(\pm 1)$ is a subset of the triangle $conv\{v_i^{\pm}(a), v_j^{\pm}(a), v_j^{\pm}(\tilde{a})\}$

$$\mathcal{A}_{k,\bar{k}}^{ij}(\pm 1) \subset conv\{v_i^{\pm}(a), v_j^{\pm}(a), v_j^{\pm}(\tilde{a})\}.$$

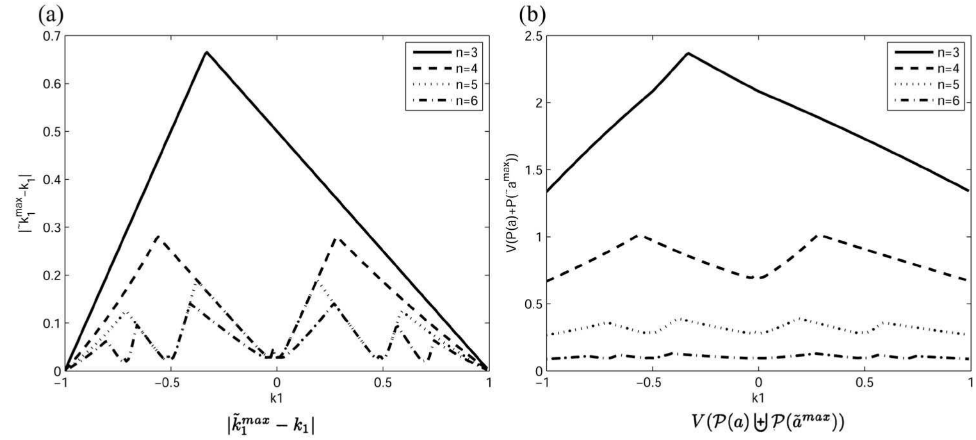

Fig. 2. The dependence of the difference $|\tilde{k}_1^{max} - k_1|$ and the volume $V(\mathcal{P}(a) \uplus \mathcal{P}(\tilde{a}^{max}))$ from the value of k_1.

The edge $conv\{v_i^{\pm}(a), v_j^{\pm}(a)\}$ of this triangle is stable as an edge of the stable reflection vector polytope $\mathcal{P}(a)$. The edges $conv\{v_i^{\pm}(a), v_j^{\pm}(\tilde{a})\}$ and $conv\{v_j^{\pm}(a), v_j^{\pm}(\tilde{a})\}$ of this triangle are stable by assumption as the edges of the stable polytope of reflection vector sets $\mathcal{P}(a) \uplus \mathcal{P}(\tilde{a})$. Thus, all the three edges of the triangle $conv\{v_i^{\pm}(a), v_j^{\pm}(a), v_j^{\pm}(\tilde{a})\}$ are stable. Therefore the triangle $conv\{v_i^{\pm}(a), v_j^{\pm}(a), v_j^{\pm}(\tilde{a})\}$ is stable and the edge $\mathcal{A}_{k,\bar{k}}^{ij}(\pm 1) = conv\{v_i^{\pm}(a), v_j^{\pm}(\bar{a})\}$ is stable. It proves the Theorem. □

Remark 1. In fact, it has been proven that the polytope of reflection vector sets $\mathcal{P}(a) \uplus \mathcal{P}(\bar{a})$ is a subset of the polytope of reflection vector sets $\mathcal{P}(a) \uplus \mathcal{P}(\tilde{a})$, if $k_1 < \bar{k}_1 < \tilde{k}_1$.

Remark 2. For fixed k_1 the volume of the polytope of reflection vector sets $\mathcal{P}(a) \uplus \mathcal{P}(\tilde{a})$ increases monotonically by increasing the difference $|\tilde{k}_1 - k_1|$.

Our aim is to find the maximal values $\tilde{k}_1^{max}$ so that the polytope of reflection vector sets $\mathcal{P}(a(k_1)) \uplus \mathcal{P}(\tilde{a}(\tilde{k}_1^{max}))$ is stable.

For $n = 2$, obviously, $\tilde{k}_1^{max} = 1$ for arbitrary k_1, $k_1 \in (-1, 1)$.

For $n \geq 3$ the maximal values $\tilde{k}_1^{max}$ can be easily found according to Theorem 2 by the bisection method. In Fig. 2a the results for $n = 3, ..., 6$ are presented.

4. VOLUME OF THE STABLE POLYTOPE OF TWO REFLECTION VECTOR SETS

The volume of the polytope of two reflection vector sets can be analytically found only for low order polynomials.

For $n = 2$, using triangulation method [20], the polytope $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ of two RV sets can be split into five simplexes

$$S_1 = [v_0, v_1^-(a), v_2^-(a)],$$
$$S_2 = [v_0, v_1^-(a), v_2^+(a)],$$
$$S_3 = [v_0, v_1^+(a), v_2^-(\tilde{a})],$$
$$S_4 = [v_0, v_1^+(a), v_2^+(a)],$$
$$S_5 = [v_0, v_2^-(a), v_2^-(\tilde{a})].$$

The first 4 simplexes have the vertices (RV) on different levels, and therefore their volume is fixed

$$V(S_j) = \frac{1}{n!} = \frac{1}{2}, \qquad j = 1, ..., 4.$$

The last simplex has the vertices (RV) on the same level and the volume of it depends on the difference $(\tilde{k}_1 - k_1)$

$$V(S_5) = \frac{2(\tilde{k}_1 - k_1)}{2!}.$$

So the total volume of $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ for $n = 2$ is

$$V(\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a}) = \sum_{j=1}^{5} V(S_j) = \frac{2(\tilde{k}_1 - k_1) + 4}{2}.$$

In Fig. 1 the polytope $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ of two RV sets with generating polynomials $a(z) = z^2 - 0.2z$ and $\tilde{a}(z) = z^2 + 0.8z$ is presented by the pentagon $(\mathbf{C}, \mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{F})$, which has considerably greater volume than the quadrangles $(\mathbf{C}, \mathbf{A}, \mathbf{D}, \mathbf{F})$ or $(\mathbf{C}, \mathbf{B}, \mathbf{D}, \mathbf{F})$. The maximal volume of the stable polytope $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ will be obtained if $(\tilde{k}_1 - k_1)$ is maximal, i.e. $k_1 = -1$ and $\tilde{k}_1 = 1$. Then the polytope $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ covers the stability domain for the second order systems (the triangle $(\mathbf{F}, \mathbf{G}, \mathbf{H})$) and the volume $V(\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})) = 4$.

For $n = 3$ and $0 \leq k_1 \leq \tilde{k}_1 \leq 1$ the following formula can be obtained via triangulation method by decoupling polytopes of the two RV sets $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})$ into simplexes with origin as the common vertex:

$$V(\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a})) = \frac{8 + (\tilde{k}_1 - k_1)(2\tilde{k}_1 + 8)}{3!}.$$

For $n > 3$ we calculated the volume $V(\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a}))$ by direct search. The first problem is to select the best value of $k_1 \in (-1, 1)$ to start from. Once k_1 is known, $\tilde{k}_1^{max}$ should be chosen such that $|\tilde{k}_1^{max} - k_1|$ is maximal, provided the stability conditions for the polytope $\mathcal{P}(a) \biguplus \mathcal{P}(\tilde{a}^{max})$ are met as well. According to Theorem 2 we can use the bisection method for calculations of the maximal value $\tilde{k}_1^{max}$.

The resulting graphs of $V(\mathcal{P}(a(k_1)) \uplus \mathcal{P}(\tilde{a}(\tilde{k}^{max})))$ for different fixed $k_1 \in (-1,1)$ and $n = 3, ..., 6$ are shown in Fig. 2b.

The peaks in the graphs of Fig. 2 clearly indicate that the volume of the polytope of two RV sets reaches maximum when the difference $|\tilde{k}_1 - k_1|$ is maximal. Therefore the bisection method can be used for maximizing $V(\mathcal{P}(a(k_1)) \uplus \mathcal{P}(\tilde{a}(\tilde{k}_1^{max})))$.

In order to compare the efficiency of the proposed method with the single reflection vector polytope, the volume of stable polytopes of different reflection vector sets for $a(z)$ with reflection coefficients $k_1 \in (-1,1)$, $k_2 = ... = k_n = 0$, are calculated [23] and presented in the second and third columns of Table 1. The fixed volume of RV polytopes for a single generating polynomial $a(z)$ is calculated by Eq. (5) and presented in the second column of Table 1. The maximal volume of stable polytopes of two RV sets is presented in the third column.

One can see that the maximal volumes of two RV sets are considerably greater than the volumes of RV polytopes for a single generating polynomial.

It is interesting to mention that the Schur invariant transformation, introduced in [12], can be used for the above stable polytopes of reflection vector sets in order to look for stable polytopes with a greater volume. The results of [12] for a single generating polynomial are represented in the third column of Table 1. The fourth column shows the outcomes of the same method, applied to stable polytopes of two RV sets.

In the last column of Table 1, the volume of stable ellipsoids, derived via optimization over linear matrix inequalities [5] (case a), are presented.

The calculation times of different approaches were compared. All calculation tests were performed using Intel Core 2 Duo with 4GB RAM in Windows Vista environment. The finding of the largest volume polytope $V(\mathcal{P}(a(k_1)))$ polytope for $n = 3$ takes 0.0001 s CPU time, the largest volume of $V(\mathcal{P}(a(k_1)) \uplus \mathcal{P}(\tilde{a}(\tilde{k}_1^{max})))$ was found in 7.9560 s (with the accuracy of $\tilde{k}_1^{max}$ equal to 0.0001) and the ellipsoid

Table 1. Comparison of volumes of the stable reflection vector polytopes for discrete-time polynomials

n	$V(\mathcal{P}(a))$	$V(\mathcal{P}(a) \uplus \mathcal{P}(\tilde{a}))$	$V_\gamma(\mathcal{P}(a))$	$V_\gamma(\mathcal{P}(a) \uplus \mathcal{P}(\tilde{a}))$	Ellipsoid
2	2.0	4.0	4.0	4.0	2.2479
3	1.3331	2.3700	1.3331	2.3700	1.4790
4	0.6665	1.0159	0.7898	1.2367	0.7770
5	0.2666	0.3911	0.2666	0.3911	0.3176
6	0.0888	0.1291	0.0982	0.1437	0.1116
7	0.0254	0.0361	0.0254	0.0361	0.0332

using LMI method [5] (case a) was built in 0.1406 s. With higher order systems, however, the results are slightly different: for $n = 7$ the calculation time for simple $V(\mathcal{P}(a(k_1)))$ is 0.0156 s, for $V(\mathcal{P}(a(k_1)) \biguplus \mathcal{P}(\tilde{a}(\tilde{k}_1^{max})))$ the calculation time is 59.4990 s and the ellipsoid calculation time is 150.0886 s.

5. CONCLUSIONS

Stable polytopes of different reflection vector sets are defined starting from the sufficient stability condition via reflection coefficients of polynomials [9] and the volume of these stable polytopes is calculated.

The volume of all stable reflection vector polytopes of a single polynomial, generated by reflection coefficients $k = [k_1 \ \ 0 \ \ \ldots \ \ 0]$, is fixed for the fixed dimension n.

The volume of the polytope $V(\mathcal{P}(k_1) \biguplus \mathcal{P}(\tilde{k}_1^{max}))$ of two reflection vector sets reaches maximum when the difference $|\tilde{k}_1 - k_1|$ is maximal. The volume of stable polytopes of two reflection vector sets is considerably greater than the volume of reflection vector polytopes of a single polynomial. That is why the stable polytopes of two reflection vector sets can be useful for robust control purposes [14,18].

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Diskreetaja süsteemide stabiilsuspiirkonna lähendid peegeldusvektorite polütoopide kaudu

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Lähtudes diskreetaja süsteemide stabiilsustingimusest karakteristliku polünoomi peegelduskoefitsientide ruumis, on leitud stabiilsed hulktahukad (polütoobid) karakteristliku polünoomi kordajate ruumis. On leitud piisavad stabiilsustingimused ühe ja mitme lähtepolünoomi korral ning vastavad stabiilsuspiirkonna lähendid peegeldusvektorite polütoopide kaudu. On arvutatud stabiilsete polütoopide ruumalad ja leitud, et mitme lähtepolünoomi puhul on stabiilse polütoobi ruumala oluliselt suurem kui ainult ühe lähtepolünoomi korral.