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Buckling of elastic beams by the Haar wavelet method

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Abstract. The Haar wavelet method is applied for solving different problems of buckling of elastic beams. Solutions are given for the following problems: (i) beams with intermediate supports, (ii) crack simulation, (iii) beams with variable cross-section, (iv) buckling and vibrations of beams on an elastic foundation. Numerical results for seven test examples are presented. It follows from the calculations that the accuracy of the results is high even in the case of a small number of calculation points. In most cases the proposed method is mathematically simpler in comparison with the conventional approaches.

Key words: buckling of elastic beams, Haar wavelets, crack simulation, elastic foundation, free vibrations.

1. INTRODUCTION

The wavelet methods have proved to be very effective for solving problems of mathematical calculus. In the last time these methods have attracted the interest of researchers of structural mechanics and many papers in this field are published. In most papers the Daubechies wavelets are applied. These wavelets are orthogonal, sufficiently smooth and have a compact support. Their shortcoming is that an explicit expression is lacking. This obstacle makes the differentiation and integration of these wavelets very complicated. For evaluation of such integrals the connection coefficients are introduced, but this complicates the course of the solution to a great extent.

Among the wavelet families, which are defined by an analytical expression, special attention deserve the Haar wavelets.They are made up of pairs of piecewise constant functions and are mathematically the simplest among all the wavelet families. A good feature of the Haar wavelets is the possibility to integrate them analytically arbitrary times. The Haar wavelets are very effective for treating singularities, since they can be interpreted as intermediate boundary conditions.

From numerous papers on buckling and vibrations of elastic beams we cite here only some, which are nearer to the topic of the present paper. Diaz et al. [1] used a hybrid scheme of Daubechies wavelets and finite element method for getting numerical solutions for Euler–Bernoulli beam. In the paper by Chen et al. [2] for that purpose the B-spline wavelets were applied. In several papers the boundary value problems have been considered. Glabisz [3] solved the multi-point boundary value problem with the aid of the Walsh wavelet packet. The solution by Zhou and Zhou [4] is based on Daubechies and Coiflet wavelets. Eliashakoff and Guede [5] considered boundary conditions for an inhomogeneous Euler–Bernoulli beam. Free vibrations of multispan beams with intermediate constraints were discussed by Lin and Chang [6]. Multispan beams of variable thickness under static loads were investigated by Xu and Zhou [7]. As to beams, resting on an elastic foundation, we would like to cite here Ayvas and Özgan [8], Chen [9] and Eliashakoff [10].

In many papers, beam structures with cracks are analysed by the wavelet methods. We refer here only to papers, in which the Haar wavelet method is applied. Wang and Deng [11] considered cracked simply supported beams subjected to a static point load. Quek et al. [12] applied for crack detecting Haar and Gabor wavelets. Gentile and Messina [13] applied vibration data for detecting open cracks. Multiresolution analysis for damage estimation in beam-like structures was applied by Kim et al. in [14].

In this paper different problems of buckling and vibrations of elastic beams are solved. The aim of the paper is mainly methodological – to demonstrate the efficiency of the Haar method. With the purpose to estimate the accuracy of the obtained results, computer simulation was carried out for the examples for which the exact solution is known. Most of these examples are taken from the classical text-book by Timoshenko [15].

The paper is organized as follows. In Section 2 formulas for calculating the integrals of the Haar wavelets are presented. The method of solution is described in Section 3. In the following sections solutions for some special problems are presented. In Section 4 beams with intermediate rigid supports are considered. A new approach for treating cracks in the beam is proposed in Section 5. Beams with variable cross-section are analysed in Section 6. In Section 7 buckling and vibrations of beams on an elastic foundation are discussed. In the last Section 8 the results of the paper are summarized.

2. HAAR WAVELETS

To make the paper self-contained, we refer to some results, obtained in [16]. Consider the interval $x \in [A, B]$, where A and B are given constants. We shall define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[A, B]$ is divided into $2M$ subintervals of equal length; the length of each subinterval is $\Delta x = (B - A)/(2M)$. Next, two other parameters are introduced: the dilatation parameter $j = 0, 1, \ldots, J$ and the translation parameter

$k = 0, 1, \dots, m-1$ (here the notation $m = 2^j$ is introduced). The wavelet number i is identified as $i = m + k + 1$.

The i-th Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } \ x \in [\xi_1(i), \xi_2(i)], \\ -1 & \text{for } \ x \in [\xi_2(i), \xi_3(i)], \\ 0 & \text{elsewhere}, \end{cases} \tag{1}$$

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta x, & \xi_2(i) &= A + (2k+1)\mu\Delta x, \\ \xi_3(i) &= A + 2(k+1)\mu\Delta x, & \mu &= M/m. \end{aligned} \tag{2}$$

The case $i = 1$ corresponds to the scaling function: $h_1(x) = 1$ for $x \in [A, B]$ and $h_1(x) = 0$ elsewhere.

In the following we need the integrals

$$p_{\alpha,i}(x) = \underbrace{\int\limits_A^x \int\limits_A^x \dots \int\limits_A^x}_{\alpha \text{ times}} h_i(t) dt^\alpha = \frac{1}{(\alpha-1)!} \int\limits_A^x (x-t)^{\alpha-1} h_i(t) dt, \tag{3}$$
$$\alpha = 1, 2, \dots, n\,, \quad i = 1, 2, \dots, 2M\,.$$

The case $\alpha = 0$ corresponds to the function $h_i(t)$.

Taking account of Eq. (1), these integrals can be calculated analytically; by doing it we obtain

$$p_{\alpha,i}(x) = \begin{cases} 0 \\ \qquad\qquad \text{for } \ x < \xi_1(i), \\ \frac{1}{\alpha!}[x - \xi_1(i)]^\alpha \\ \qquad\qquad \text{for } \ x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!}\left\{[x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha\right\} \\ \qquad\qquad \text{for } \ x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!}\left\{[x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha + [x - \xi_3(i)]^\alpha\right\} \\ \qquad\qquad \text{for } \ x > \xi_3(i). \end{cases} \tag{4}$$

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi_1 = A$, $\xi_2 = \xi_3 = B$ and

$$p_{\alpha,1}(x) = \frac{1}{\alpha!}(x - A)^\alpha. \tag{5}$$

For solving boundary value problems we need the values $p_{\alpha,i}(B)$, which can be calculated from Eq. (4). In special cases, $\alpha = 1$ or $\alpha = 2$, we find

$$q_1(i) = p_{1,i}(B) = \begin{cases} B - A & \text{for } i = 1, \\ 0 & \text{for } i \neq 1, \end{cases} \tag{6}$$

and

$$q_2(i) = p_{2,i}(B) = \begin{cases} 0.5(B-A)^2 & \text{for } i = 1, \\ 0.25\dfrac{(B-A)^2}{m^2} & \text{for } i \neq 1. \end{cases} \tag{7}$$

3. PROBLEM STATEMENT AND THE METHOD OF SOLUTION

Consider buckling of an elastic beam under axial compressive load P. If the load obtains the critical value P_{cr}, the beam buckles. The deflection curve of the buckled beam is determined from the governing equation

$$EI(\tilde{x})\mathrm{d}^2 w/d\tilde{x}^2 = M(\tilde{x}),\ \tilde{x} \in [0, L]. \tag{8}$$

Here E is the Young modulus, I is the moment of inertia of the cross-section and M denotes the bending moment at the cross-section $\tilde{x}$. In the following, two variants of the boundary conditions are considered:
(i) beam with simply supported ends, here $M = -Pw$,
(ii) cantilever beam with clamped ends, $M = P(\delta - w),\ \delta = w(L)$.

Let us change the variables

$$x = \tilde{x}/L,\ \lambda = PL^2/EI(\tilde{x}). \tag{9}$$

Now Eq. (6) obtains the form:
(i) for the simply supported beam

$$w'' + \lambda w = 0, \quad w(0) = w(1) = 0, \tag{10}$$

(ii) for the cantilever beam

$$w'' + \lambda w = \lambda\delta, \quad w(0) = w'(0) = 0, \quad w(1) = \delta. \tag{11}$$

Here and in the following primes denote differentiation with regard to x.

According to the Haar wavelet method, the solution of (10)–(11) is expressed in the form

$$w''(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{12}$$

Integrating this equation we obtain

$$\begin{aligned} w'(x) &= \textstyle\sum_{i=1}^{2M} a_i p_{1i}(x) + C_1, \\ w(x) &= \textstyle\sum_{i=1}^{2M} a_i p_{2i}(x) + C_1 x + C_2. \end{aligned} \tag{13}$$

Here a_i are the wavelet coefficients; the quantities h_i, p_{1i}, p_{2i} are evaluated from Eqs (1), (4), (5), respectively. The integration constants are calculated from the boundary conditions, and we obtain

(i) $C_1 = -\sum_{i=1}^{2M} a_i q_2(i),\ C_2 = 0$ for the simply supported beam,
(ii) $C_1 = C_2 = 0$ for the cantilever beam.

All these results are substituted into Eqs (10)–(11) and the obtained differential equation is satisfied in the collocation points. By doing this we get the system of equations

$$\sum_{i=1}^{2M} a_i\{H(i,l) + \lambda[P_2(i,l) + \beta_1 Q(i,l)]\} = \beta_2 F(l),\ \ l = 1, 2, \ldots, 2M. \quad (14)$$

Here the notations $Q(i,l) = -q_2(i)x_l,\ F(l) = \lambda\delta$ were introduced. For a simply supported beam we have $\beta_1 = 1,\ \beta_2 = 0$; for the cantilever beam $\beta_1 = 0,\ \beta_2 = 1$. It is advantageous to put Eq. (14) into the matrix form

$$aR = \beta_2 F, \quad (15)$$

where

$$R = H - \lambda(P_2 + \beta_1 Q). \quad (16)$$

The system (15) is linear and homogeneous with regard to the variables a_i and δ. For getting a non-trivial solution, the determinant of this system must be zero. From this requirement the critical load parameter λ_{cr} is evaluated.

To obtain the deflection curve $w = w(x)$, we must specify the deflection in some cross-section $x = x*$. According to Eq. (13) we get a complementary equation

$$w(x*) = \sum_{i=1}^{2M} a_i p_{2i}(x*) + C_1 x * + C_2, \quad (17)$$

which is incorporated into the system (15). The augmented system is now non-homogeneous and has a non-trivial solution for a_i and δ. The function $w = w(x)$ is calculated from Eq. (13).

4. BEAM ON INTERMEDIATE SUPPORTS

Consider an axially compressed beam with n rigid supports at the cross-sections $y_\alpha \in (0,1), \alpha = 1, 2, \ldots, n$. The conditions $w(y_\alpha) = 0$ can be interpreted as intermediate boundary conditions for the system (15) and in view of Eq. (17) we have

$$\sum_{i=1}^{2M} a_i p_{2i}(y_\alpha) + C_1 y_\alpha + C_2 = 0, \quad (18)$$

where the quantities $p_{2i}(y_\alpha)$ are calculated from Eqs (4) and (5). Adding these equations to the system (15), we get an augmented system of $2M + n$ equations. For getting non-trivial solutions, the rank of this system must be less than $2M$.

For realizing this requirement, we choose from the augmented matrix a submatrix of order $2M$ and vary the load parameter so that the determinant of this submatrix turns to zero. Since the system is homogeneous then all the $2M$-order determinants, made up from the augmented system, are also zero. Consequently, the calculated value for λ is really critical.

Example 1. Consider a simply supported beam on two rigid intermediate supports with the locations $y_1 = 1/3,\ y_2 = 2/3$. The exact solution of this problem is $\lambda_{\rm cr} = 9\pi^2 = 88.82$ ([15], Section 19). Our computations gave $\lambda_{\rm cr} = 89.14$ for $J = 4$ (with the error of 0.3%) and $\lambda_{\rm cr} = 88.91$ for $J = 5$ (error 0.09%). For putting together the deflection curve it is assumed that $w'(0) = 1$. In view of Eq. (13) this condition obtains the form

$$\sum_{i=1}^{2M} a_i q_2(i) = -1. \tag{19}$$

According to the conventional method of solution, the multi-span beams are divided into spans between the two consequent supports. The governing equation is integrated separately for each span. The integration constants are calculated from the continuity conditions at the supports [6,15]. If the number of intermediate supports is greater than 2, this approach may turn out to be very troublesome. Our method is much more simple since we treat the beam as a whole (not dividing it into parts).

5. CRACK SIMULATION

Let us assume that the moment of inertia of the cross-section is $I = I_0 = \mathrm{const}$ except some discrete points $x = y_\alpha,\ \alpha = 1, 2, \ldots, n$, in which $I = I_\alpha < I_0$. Such a situation can be interpreted as a damaged beam, which has n cracks of infinitesimal width at the locations y_α. Since the bending moment must be continuous in sections $x = y_\alpha$, then in view of Eq. (8) we find

$$w''(y_\alpha - 0) = \gamma_\alpha w''(y_\alpha),\ \ \gamma_\alpha = I_\alpha/I_0. \tag{20}$$

Equations (10) to (11) obtain in the sections $x = y_\alpha$ the form

$$\gamma(\alpha)w''(y_\alpha) + \lambda w(y_\alpha) = \lambda\delta,\ \lambda = PL^2/(EI_0). \tag{21}$$

In the case of the simply supported beam we shall take $\delta = 0$.

Equation (15) remain valid also for $x = y_\alpha$ if we take

$$R_\alpha(i, y_\alpha) = \gamma_\alpha h_i(y_\alpha) - \lambda[p_{2i}(y_\alpha) - \beta_1 q_2(i) y_\alpha]. \tag{22}$$

The values $h_i(y_\alpha)$ and $p_{2i}(y_\alpha)$ are calculated from Eqs (1), (4) and (5). Incorporating Eq. (22) into Eq. (15) we get a system of $2M + n$ equations, from which the critical parameter λ_{cr} can be calculated. The following course of solution proceeds as shown in Section 4.

Example 2. Computer simulation was carried out for a simply supported beam with a crack at $x = 0.5$. The results for $J = 4$ are presented in Table 1. The symbol $\Delta = \lambda_{\text{crack}}/\lambda$ shows the relative reduction of the buckling capacity for the cracked beam. The functions $w(x),\ w'(x)$ and $w''(x)$ for some values of γ are plotted in Fig. 1.

Table 1. Reduction of the buckling load of a simply supported beam with a crack at $x = 0.5$

γ	0.8	0.6	0.4	0.2	0.1	0.05	0.01
λ_{cr}	9.72	9.47	9.01	7.83	6.14	4.23	1.18
Δ	0.99	0.96	0.91	0.79	0.62	0.43	0.11

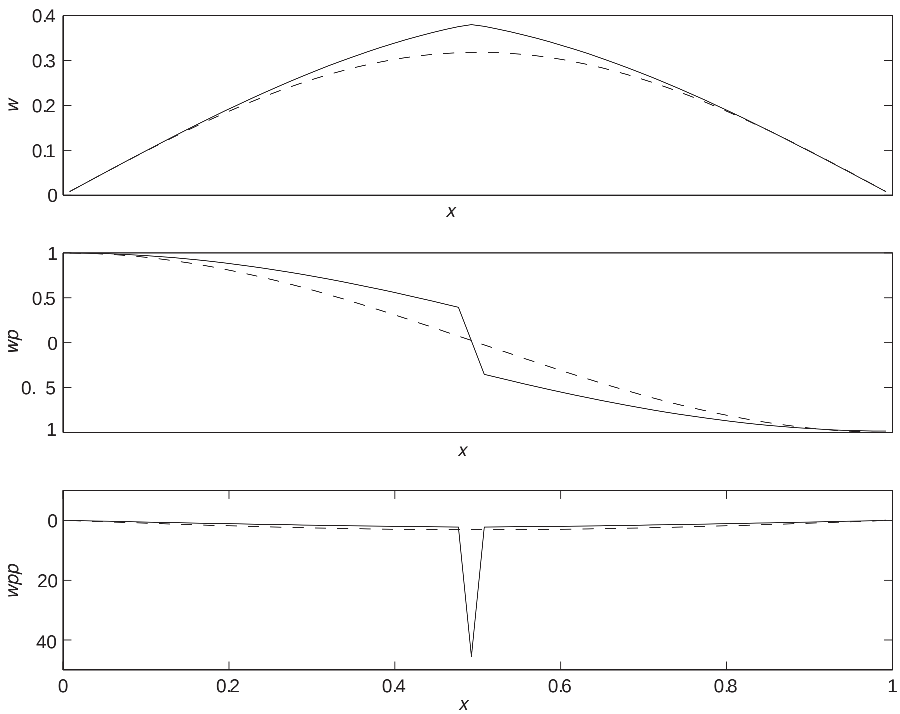

Fig. 1. Deflection w, slope wp and curvature wpp of a simply supported beam with a crack at $x = 0.5$; – damaged beam, - - undamaged beam.

Example 3. Karaagac et al. [17] and Skrinar [18] investigated buckling of a cantilever beam with rectangular cross-section. The beam has a crack, located at $y \in (0, 1)$. The crack's depth is $d = 1 - h/h_0$, where h_0 is the beam's thickness, h is the thickness of the damaged section. The crack is modelled by a linear massless rotational spring, connecting the uncracked parts of the beam. The governing equation is integrated separately for both uncracked segments. For the connection between both parts, the continuity of the displacement, bending moment and shear force is imposed.

It is interesting to compare these results with our outcome. We have carried out computations for three variants of the crack location: $y_1 = 0.2$, $y_2 = 0.5$ and $y_3 = 0.8$. For the crack depth the value $d = 0.5$ was taken. Since the two crack parameters are related according to the formula $\gamma = (1 - d)^3$, then in the present case $\gamma = 0.125$. An overview of the obtained results is presented in Table 2. The symbol Δ_w denotes our wavelet results. Experimental data from the paper by Karaagac et al. [17] are marked by $\Delta_{\rm exp}$. The coefficients Δ_1 correspond to a FEM solution from the same paper. Data in the last two columns are taken from the paper by Skrinar [18]. For calculating Δ_2, the COSMOS 2D FEM program was applied (here 20000 8-noded finite elements with more than 122 000 degrees of freedom were used). The coefficients Δ_3 are calculated by the analytical model of Skrinar.

It follows from Table 2 that the results obtained by the Haar wavelet method are in satisfactory accordance with the data of other authors. Our method is substantially simpler, since we do not need to introduce rotational springs and divide the beam into parts between the damaged sections. Difference is also in continuity conditions at the cracked sections. In the conventional approach the gradient w' is regarded as discontinuous. In our solution it is continuous (discontinuous is the curvature). This variant of the continuity conditions seems to us more logical, since if the first derivative w' is discontinuous then the second derivative is indeterminate. We consider our results about crack simulation as preliminary – this model should be tested by solving more problems about cracked structures. This is the object of further research.

Table 2. Reduction of the buckling load of a cracked cantilever beam, comparison of theoretical and experimental results

y	Δ_w	$\Delta_{\rm exp}$	Δ_1	Δ_2	Δ_3
0.2	0.832	0.789	0.904	0.900	0.891
0.5	0.900	0.915	0.978	0.933	0.937
0.8	0.980	0.978	0.989	0.985	0.987

6. BUCKLING OF BEAMS OF VARIABLE CROSS-SECTION

Consider the case where the moment of inertia $I(x)$ is a given piecewise constant function of the coordinate x. Let us assume $I(0) \neq 0$ and denote $\gamma(x) = I(x)/I(0)$. In view of Eq. (9) we have $\lambda(x) = \lambda_0/\gamma(x)$, where $\lambda_0 = PL^2/[EI(0)]$. Equations (15)–(16) remain valid if we replace there H by $\gamma(x_l)H$. If the function $\gamma(x)$ has points of discontinuity $x = y_\alpha,\ \alpha = 1, 2, \ldots, n$, then in these points the condition (21) must be fulfilled. We have to find such a value of λ_0 at which the buckling begins.

In the textbook by Timoshenko ([15], Section 26), buckling of cantilever beams of variable cross-section is discussed. To compare our results with the Timoshenko data we take

$$\gamma(x) = \gamma_1[1 + (\gamma_1^{-1/n} - 1)(1 - x)]^n,\ \gamma_1 = I(1)/I(0),\ n = 1, 2, 3, \ldots. \quad (23)$$

If $n = 2$, the exact solution of the problem exists. For $n > 2$ the solution can be found with the aid of Bessel functions. Timoshenko's numerical results for $n = 2$ and $n = 4$ are presented in Tables 11–12 of his book [15]. We have carried out computations for $J = 5, \gamma_1 = 0.4$. In the case $n = 2$ we have $(\lambda_0)_{\rm cr} = 1.906$ (Timoshenko's solution gives 1.904). For $n = 4$ both solutions give he same result $(\lambda_0)_{\rm cr} = 1.870$.

Example 4. The proposed method of solution is applicable also for stepped beams. Consider a two-stepped beam with $y_1 = 0.4$ and $\gamma(x) = 1$ for $x \in [0, y_1]$, $\gamma(x) = 0.6$ for $x \in [y_1, 1]$. Computer simulation for $J = 4$ gave $(\lambda_0)_{\rm cr} = 2.056$. The same problem was solved by Timoshenko ([15], Section 25), who obtained $(\lambda_0)_{\rm cr} = 2.048$; difference is 0.39%. Computations were carried out also for the three-stepped beam for which $y_1 = 1/3,\ y_2 = 2/3$ and $\gamma(x) = 1$ for $x \in [0, y_1],\ \gamma(x) = 0.7$ for $x \in [y_1, y_2],\ \gamma(x) = 0.4$ for $x \in [y_2, 1]$. The critical load parameter value $(\lambda_0) = 1.988$ was obtained. Again we can state that our solution is essentially simpler to be compared with the traditional approaches since we treat the beam as a whole and do not divide it into parts for which $I(x) = \text{const}$.

7. BUCKLING OF BEAMS ON ELASTIC FOUNDATION

The governing equation for buckling and vibrations of an axially compressed beam on elastic foundation is

$$EI\frac{\partial^4 w}{\partial \tilde{x}^4} + (P - \tilde{K}_2)\frac{\partial^2 w}{\partial \tilde{x}^2} + \tilde{K}_1(\tilde{x})w = -\rho A\frac{\partial^2 w}{\partial t^2}. \quad (24)$$

Here $I = \text{const}$, $\tilde{K}_1(\tilde{x})$ is the variable coefficient of Winkler foundation, $\tilde{K}_2$ is the Pasternak foundation coefficient, ρ is the material density of the beam and A is the cross-sectional area.

Introducing dimensionless quantities

$$x = \tilde{x}/L,\ \lambda = PL^2/(EI),\ K_1 = \tilde{K}_1 L^4/(EI),\ \tilde{K}_= L^2/(EI),\ \mu = \rho A L^4/(EI) \tag{25}$$

we can rewrite Eq. (24) in the dimensionless form

$$w^{IV} + (\lambda - K_2)w'' + K_1(x)w = -\mu w^{\cdot\cdot}. \tag{26}$$

Solution of this equation is sought in the form

$$w(x,t) = W(x)T(t). \tag{27}$$

Introducing Eq. (27) into Eq. (26) and separating the variables, we obtain

$$T(t) = A\cos(\omega t) + B\sin(\omega t), \tag{28}$$

$$W^{IV} + (\lambda - K_2)W'' + [K_1(x) - \mu\omega^2]W = 0, \tag{29}$$

where ω denotes the frequency of the beam vibrations.

The wavelet solution of Eq. (29) is taken in the form

$$W^{IV} = aH. \tag{30}$$

By multiple integration of this equation we find

$$\begin{aligned} W''' &= aP_1 + C_1 E, \\ W'' &= aP_2 = C_1 x + C_2 E, \\ W' &= aP_3 + (1/2)C_1 x.^2 + C_2 x + C_3 E, \\ W &= aP_4 + (1/6)C_1 x.^3 + (1/2)C_2 x.^2 + C_3 x + C_4 E. \end{aligned} \tag{31}$$

Here $E = [1, 1, \ldots, 1]$ and the decimal point denotes element-by-element multiplication. The integration constants C_1, C_2, C_3 and C_4 are calculated from the boundary conditions.

(i) For a simply supported beam the boundary conditions are $W(0) = W''(0) = W(1) = W''(1) = 0$ and we obtain

$$C_1 = -aq_2,\ C_2 = C_4 = 0,\ C_3 = -a[q_4 - 0.5q_2]. \tag{32}$$

(ii) Satisfying the boundary conditions $W(0) = W'(0) = W''(1) = W'''(1) = 0$ for the cantilever beam, we get

$$C_1 = -aq_1,\ C_2 = a(q2 - q1),\ C_3 = C_4 = 0. \tag{33}$$

The vectors $q_\alpha(i) = p_{\alpha i}(1), \alpha = 1, 2, 3, 4$ are calculated according to Eqs (4) to (6). All these results are introduced into Eq. (29) and the obtained equation is satisfied at the collocation points. The outcome can be presented in the matrix form as $aR = 0$, where

$$R(i,l) = H(i,l) + (\lambda - K_2)[P_2(i,l) + x_l q_2(i)] \\ + [K_1(l) - \mu\omega^2]\{P_4(i,l) + x(l)(q_2(i)/6 - q_4(i) - x_l^3 q_2(i)/6\} \quad (34)$$

for the simply supported beam and

$$R(i,l) = H(i,l) + (\lambda - K_2)[P_2(i,l) - x_l q_2(i) + q_2(i) - q_1(i)] \\ + [K_1(l) - \mu\omega^2]\{P_4(i,l) + x(l)^2[-x_l q_1(i) + 3q_2(i) - q_1(i)]/6\} \quad (35)$$

for the cantilever beam.

Such critical values for λ (or ω), for which the determinant $|R|$ is zero, must be found.

Example 5. Calculate the buckling load for an axially compressed simply supported beam on Winkler's foundation. This problem was discussed by Timoshenko ([15], Section 21). Timoshenko gave for the critical load the formula, which in our notations has the form

$$\lambda_{\mathrm{cr}} = \pi^2(m + \frac{K_1}{m^2\pi^4}), \quad K_1 = \mathrm{const.} \quad (36)$$

Here m is a positive integer.

For $K_1 = 100$ it follows from Eq. (36) that $\lambda_{\mathrm{cr}} = 20.003$ and $m = 1$; our wavelet solution gives $\lambda_{\mathrm{cr}} = 20.007$. For $K_1 = 400$ Timoshenko's solution is $\lambda_{\mathrm{cr}} = 49.61,\ m = 2$; our solution gives $\lambda_{\mathrm{cr}} = 49.62$. So we see that accordance of the two solutions is rather good. The displacement curves for both solutions are plotted in Fig. 2.

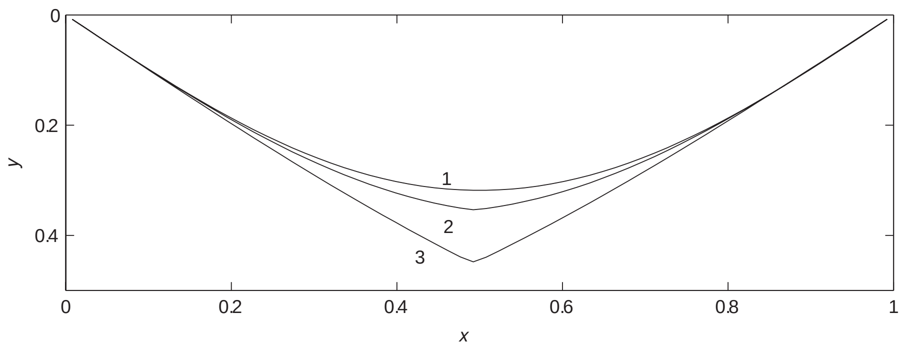

Fig. 2. Buckling of a simply supported beam on Winkler's foundation, – solution for $\gamma = 100$, - - solution for $\gamma = 400$.

Table 3. Critical frequencies of a vibrating cantilever beam on Winkler's foundation.

ω	Exact	Wavelet	DQEM1	DQEM2
ω_1	3.655	3.656	3.655	3.655
ω_2	22.056	22.063	22.056	22.057
ω_3	61.706	61.731	61.668	61.706

Example 6. Chen [9] considered free vibrations of a cantilever beam on Winkler's foundation. First three critical frequencies $\omega_1,\ \omega_2,\ \omega_3$ were calculated by the differential quadrature element method (DQEM). Computer simulation was carried out for $\lambda = 0,\ K_2 = 0,\ K_1 = 1,\ \mu = 1$ (our notations).

We have solved the same problem by the Haar wavelet method for $J = 4$ (32 collocation points). All these results are shown in Table 3. Since in the present case Eq. (29) is a linear equation with constant coefficients, it is not difficult to put together an exact solution of the problem. For conciseness sake the course of this solution is not shown here; the calculated critical frequencies are indicated in the second column of Table 3. In the third column our wavelet results are presented. The data in the fourth and fifth column are taken from Chen's paper [9]: the data DQEM1 correspond to five-node model with 8 elements; for DQEM2 the nine-mode model with 8 elements was used. A good accordance of all these results can be stated.

8. CONCLUSIONS

The Haar wavelet method exhibits several advantageous features.

(i) High accuracy is obtained already for a small number of grid points.

(ii) Possibility of implementation of standard algorithms. For calculation the integrals of the wavelet functions (3), universal subprograms can be put together. Another time-consuming operation is the solving of high-order systems of linear equations and calculating high-order determinants; here the matrix programs of MATLAB are very effective.

(iii) The method is very convenient for solving boundary value problems since the boundary conditions are taken care of automatically.

(iv) Singularities can be treated as intermediate boundary conditions, this circumstance to a great extent simplifies the solution. In the present paper this approach was applied for beams with intermediate supports, for stepped beams and for crack simulation.

(v) The obtained solutions are mostly simpler compared with other known methods.

The aim of this paper was to demonstrate positive features of the Haar wavelet method for treating buckling problems of elastic beams. For pedagogical reasons, only simple problems were considered, but the proposed technique is applicable without essential changes also for more complicated problems.

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Elastsete talade nõtke uurimine Haari lainikute meetodil

Ülo Lepik

Haari lainikud on osutunud väga efektiivseteks mitmesuguste diferentsiaal- ja integraalvõrrandite lahendamisel. Käesoleva töö eesmärgiks on rakendada seda metoodikat elastsusteooria ülesannetele. On vaadeldud mitmeid elastsete varraste nõtkeülesandeid (mitme toega vardad, pragude modelleerimine, muutuva paksusega vardad, vardad elastsel alusel). Kasutatud meetodi efektiivsuse hindamiseks on näiteülesanneteks valitud klassikalised probleemid, mille puhul on täpne lahend teada. Tulemuste analüüs näitab, et paljudel juhtudel on Haari lainikute meetod traditsioonilistest meetoditest oluliselt lihtsam ja võimaldab saavutada vajalikku täpsust juba väheste kollokatsioonipunktide korral. Soovitatud meetod on eriti efektiivne juhul, kui konstruktsioonil on singulaarsusi (astmeliselt muutuv paksus, praod jne).