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Complete bifurcation analysis of driven damped pendulum systems

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Abstract. The pendulum systems are widely used in engineering, but their qualitative behaviour has not been investigated enough. Therefore the aim of this work is to study new non-linear effects in three driven damped pendulum systems, which are sufficiently close to the real models used in dynamics of machines and mechanisms. In this paper the existence of new bifurcation groups, rare attractors and chaotic regimes in the driven damped pendulum systems is shown.

Key words: pendulum systems, damping, complete bifurcation analysis, complete bifurcation groups, rare attractors, chaos, domains of attraction.

1. INTRODUCTION

Recent research in non-linear dynamics shows, that the so-called rare attractors (RA) exist in all typical and well studied models [1–9], but are unnoticed by the traditional methods of analysis. The systematic research of rare attractors is based on the method of complete bifurcation groups (MCBG) [6]. This method allows conducting more complete global analysis of the dynamical systems. The main idea of the approach is periodic branch continuation along stable and unstable solutions. The method is based on the ideas of Poincaré, Birkhoff, Andronov and others [6,7]. It is shown that the MCBG allows to find important rare attractors and new bifurcation groups in different non-linear models. The main features of this method are illustrated in this work by three driven damped pendulum systems.

Our aim is to build complete bifurcation diagrams and to find unknown rare regular and chaotic attractors using complete bifurcation analysis for some important parameters of the model: the amplitudes and the frequency of excitation. For complete global bifurcation analysis we have used the MCBG, Poincaré mappings, basins of attraction, etc.

The main results of this work are presented by complete bifurcation diagrams for variable parameters of the driven damped pendulum systems. We consider three pendulum models: a) model with an additional linear restoring moment and with the periodically vibrating in both directions point of suspension, b) model with a linear restoring moment and with the external periodic moment of excitation, c) model with a sliding mass and with the external periodic moment of excitation. By building the complete bifurcation diagrams with stable and unstable periodic solutions, we have found different new bifurcation groups with their own rare regular and chaotic attractors. All results were obtained numerically, using software NLO [1] and SPRING [2], created at Riga Technical University.

2. MODELS AND RESULTS

The first pendulum model is shown in Fig. 1a. The system has additional linear restoring moment with the harmonically vibrating in both directions point of suspension. Restoring moment and backbone curves for the system are shown in Figs. 1b,d. The system has three equilibrium positions (Fig. 1c).

The first mathematical pendulum model is described by the following equation of motion:

$$mL^2\ddot{\varphi} + b\dot{\varphi} + c\varphi + mL(\mu - A_2\omega^2\cos\omega t)\sin\varphi + mLA_1\omega^2\sin\omega t\cos\varphi = 0, \qquad (1)$$

where φ is the angle of rotation, read-out from a vertical line, $\dot{\varphi}$ is angular velocity of the pendulum $\dot{\varphi} = \mathrm{d}\varphi/\mathrm{d}t$, t is time, m is mass, L is the length of the

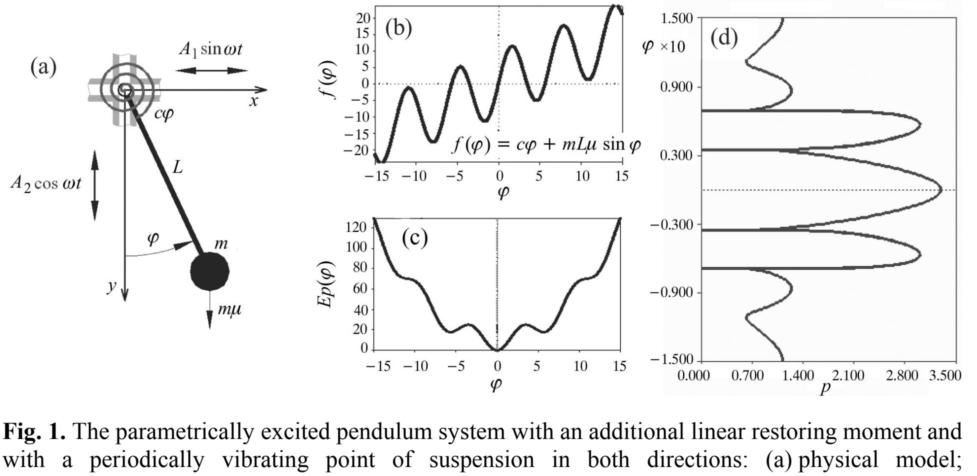

Fig. 1. The parametrically excited pendulum system with an additional linear restoring moment and with a periodically vibrating point of suspension in both directions: (a) physical model; (b) restoring moment; (c) potential well; (d) backbone curve.

pendulum, μ is the gravitation constant, b is the linear damping coefficient, c is the linear stiffness coefficient, A_1, A_2 and ω are oscillation amplitudes and frequency of the point of suspension in the horizontal and a vertical direction. The variable parameters of the pendulum system are horizontal and vertical oscillation amplitudes A_1 and A_2 of the suspension point. Other parameters are fixed: $m=1$, $L=1$, $b=0.2$, $c=1$, $\mu=9.81$, $\omega=1.5$.

Similar models have been examined in papers [10–15]. But the structures of these pendulum-like models are different, because of the difference in excitations of the pendulum support and in restoring characteristics. In our work, instead of the elliptical excitation, etc, we consider the periodically vibrating in both directions point of suspension and the external periodic moment of excitation. Moreover, in this paper we investigate two models with an additional linear restoring moment and a third model with two degrees of freedom and with a sliding mass. The main difference is in the aim of these investigations. Our aim was to build complete bifurcation diagrams and to find new bifurcation groups and unknown rare regular and chaotic attractors using complete bifurcation analysis in these pendulum systems. Some results of complete bifurcation analysis for the first pendulum model were presented in several previous papers [16,17]. According to [15], there is a difference in the used methods. Our method of complete bifurcation groups is intended for the construction of bifurcation diagrams, which, as a rule, follow up the stable regimes only (the so-called brute-force). The basic feature of this method is following up of all stable and unstable regimes in the parameter space together with their stability characteristics and the origin of unstable periodic infinity [6,7]. As was mentioned above, the method is based on the ideas of Poincaré, Birkhoff and Andronov and used together with Poincaré mappings, mappings from a line and from a contour, basins of attraction, etc.

The results of bifurcation analysis of the model (1) are represented in Figs. 2 to 5. Five different 1T bifurcation groups and one 2T bifurcation group have been found (Fig. 2). The symbols 1T and 2T mean the characteristic of the period of free oscillations dependent on period T_ω of excitation force: $T=nT_\omega$, $n=1, 2, 3, \dots$. If $n>2$, there are subharmonic oscillations.

Two of these groups are topologically similar and have rare attractors of a tip kind $P1_1$ RA and $P1_3$ RA. The symbol P1 means the characteristic of the period of oscillation regime. For this case it is a regime of period 1. Subscripts 1, 2, 3 etc mean the classification numbers of bifurcation groups, because of the existence of many bifurcation groups with the same period of oscillations. The stable solutions are plotted by solid lines and unstable – by red lines .

Two period one brunches near $A_2=4$ are not completed, because of problems of sufficiently high instability with maximal value less than 2500. Other three 1T bifurcation groups have their own rare attractors $P1_4$ RA and $P1_5$ RA, which are stable in small parameter regions.

Some cross-sections $(A_2=\text{const})$ of bifurcation diagrams with dynamical characteristics from Fig. 2 are represented in Figs. 3 and 4. All attractors are of the tip kind so each of them has not only periodic attractors, but also chaotic attractors as well. The examples of coexistence of period 1 (P1) stable solutions and P1 RA rare attractors for three cross-sections $A_2 = 0.44$, $A_2 = 0.526$ and $A_2 = 3.44$ on bifurcation diagram (Fig. 2a) with the time histories and phase projections are shown in Fig. 4. Oscillation amplitudes of rare attractors in some cases are tenfold bigger than oscillating amplitudes of stable P1 regimes. The examined system has also another bifurcation group of higher order with rare attractors, for example, 2T bifurcation group with P2 RA rare attractor with large oscillation amplitudes.

Each bifurcation group has its own unstable periodic infinity (UPI) [[1–5]] with corresponding chaotic attractors. UPI is a sub-bifurcation group based on Poincaré and Birkhoff ideas with infinite unstable periodic solutions *n*T, 2*n*T, 4*n*T, … [[6]]. The existence of the UPI due to the complete cascade of *n*T-period doubling and the crisis is a necessary part of the typical bifurcation group with chaotic behaviour. It is a well-known fact that the presence of the UPI characterizes the parameter region with chaotic attractors and/or chaotic transients (see Fig. 2). It is known that the system may have several different UPIs simultaneously, and so the resulting chaotic behaviour depends in this case on each bifurcation group with UPI [[6]]. The example of globally stable chaotic attractor for cross-section with parameters $A_1 = 0.5$, $A_2 = 4.9$, obtained using the contour mapping [[1–9]], is shown in Fig. 3a.

The system also has other *n*T subharmonic bifurcation groups with $n = 3$, not shown.

Comparison of two different methods for building domains of attraction of tip kind rare attractor $P1_1$ RA is represented in Fig. 5. These methods show the same results of building domains of attraction.

The second mathematical pendulum model, shown in Fig. 6a, is described by the following equation of motion:

$$\ddot{\varphi} + b\dot{\varphi} + (a\varphi + a_1 \sin 2\pi\varphi) = h_1 \cos \omega t, \tag{2}$$

where φ is the angle, read out from a vertical, $\dot{\varphi}$ is the angular velocity of the pendulum, a is the linear stiffness coefficient, a_1 is a coefficient, which includes pendulum length and gravitational constant, and $h_1 \cos \omega t$ is harmonical moment, enclosed at a point of support. The variable parameter of the pendulum system is frequency ω of the external periodic excited moment. Other parameters are fixed: $b = 0.1$, $a = 1$, $a_1 = 0.1$, $h_1 = 1$.

For the given system with one equilibrium position the complete bifurcation analysis was made. Results of the analysis are shown in Fig. 7. Special feature of this system is the unexpected isolated P1 isle, amplitudes of vibrations of which are much greater than ones of the usual P1 regime. Also for these three attractors P1 the domains of attraction were constructed (Fig. 8) for cross-section $\omega = 0.347$ of the bifurcation diagram in Fig. 7.

Equations of motion for the third mathematical pendulum model with a sliding mass and with the external periodic excited moment (Fig. 9) are

$$\begin{cases} [m_1 l^2 + m_2(l - l_0 + y)^2]\ddot{\varphi} + b_1\dot{\varphi} + 2m_2(l - l_0 + y)\dot{\varphi}\dot{y} + m_1\mu l \sin\varphi \\ \qquad + m_2\mu(l - l_0 + y)\sin\varphi = M(\omega_1 t), \\ m_2\ddot{y} + b_2\dot{y} + c_2 y - m_2(l - l_0 + y)\dot{\varphi}^2 + m_2\mu(1 - \cos\varphi) = 0, \end{cases} \tag{3}$$

where y is the displacement of the sliding mass, read out from a quiescent state, $\dot{y}$ is velocity of the sliding mass, m_1 is the pendulum mass, l is length of the pendulum, m_2 is second mass, l_0 is the quiescent state of the sliding mass, μ is gravitational constant, b_1 and b_2 are linear damping coefficients of the pendulum and the sliding mass, c_2 is the linear stiffness coefficient of the sliding mass, $M(\omega_1 t) = h_1 \cos\omega_1 t$ is the external periodically excited moment and h_1 and ω_1 are the amplitude and frequency of excitation. The variable parameters of the pendulum system are amplitude h_1 and frequency ω_1 of excitation. Other parameters are fixed: $m_1 = 1$, $m_2 = 0.1$, $l = 1$, $l_0 = 0.25$, $b_1 = 0.2$, $b_2 = 0.1$, $c_2 = 2$, $\mu = 10$.

Similar models have been studied in [[14]]. Results for this system are represented in Figs. 10 and 11. One 1T and one 5T bifurcation group with Andronov–Hopf bifurcations, several symmetry breakings, period doublings and rare attractor P5 RA (Fig. 10a) have been found in the third model.

In Fig. 10b there are 1T and 2T bifurcation groups with several symmetry breakings, period doublings, folds and tip type rare attractors. Global chaos ChA-1 has been found (Fig. 11) in this pendulum system using the Poincaré mapping Cm 4 × 50Q × (400–500)T from a contour. Figures 10 and 11 show that the method of complete bifurcation groups allows finding new unnoticed before regimes also in the system with two degrees of freedom. Thus, the application of this method for global analysis of forced oscillations is also possible for systems with several degrees of freedom.

3. CONCLUSIONS

The pendulum systems are widely used in engineering, but their qualitative behaviour has not been investigated enough. Therefore in this work the new non-linear effects in three driven damped pendulum systems, which are sufficiently close to the real models used in dynamics of the machines and mechanisms, were shown. These results were obtained using the method of complete bifurcation groups. Only the method of complete bifurcation groups allows finding rare periodic and chaotic regimes systematically. These regimes can lead to small breakages of machines and mechanisms and to global technical catastrophes, because they are unexpected and usually have large amplitudes.

Some of the obtained new effects can be used for the parametric stabilization of unstable oscillations in technological processes. Authors hope to attract attention of scientists and engineers to solving the important problems of non-linear oscillations analysis by the MCBG. Rare dangerous or useful attractors may find application in the pendulum-like dynamical systems.

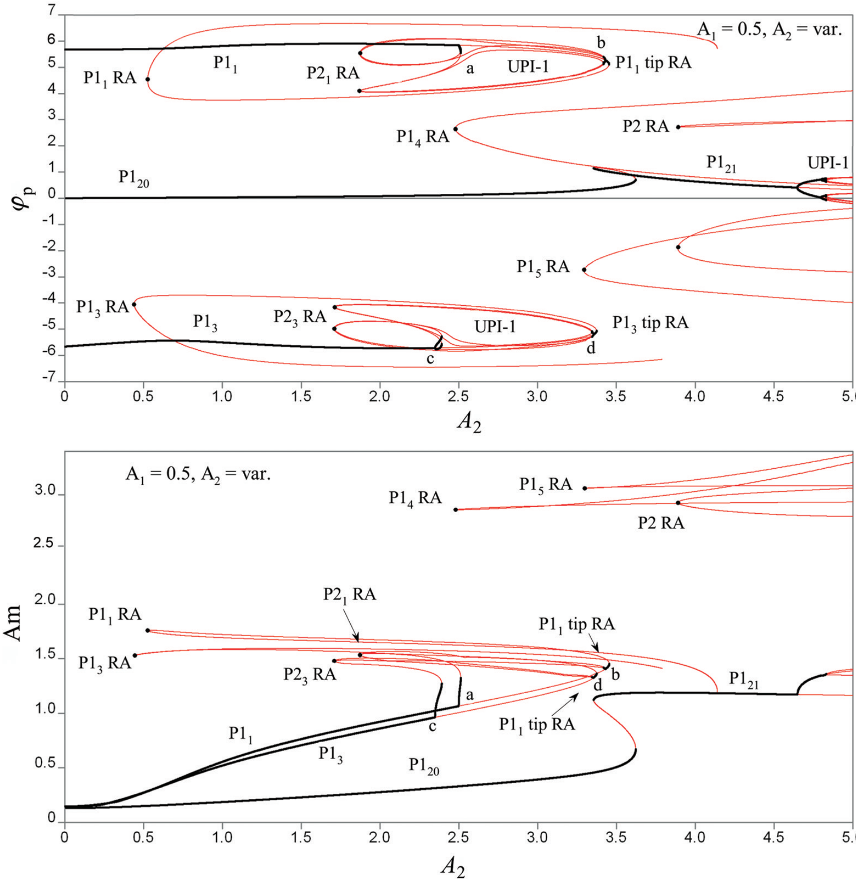

Fig. 2. The driven damped pendulum system (Eq. (1)) with the periodically vibrating point of suspension in both directions. Bifurcation diagrams of the fixed periodic points of the angle of rotation φ_p and amplitude of rotation Am versus vertical oscillation amplitude A_2. There are five 1T and one 2T bifurcation groups. The pendulum system has many rare attractors of different kinds.

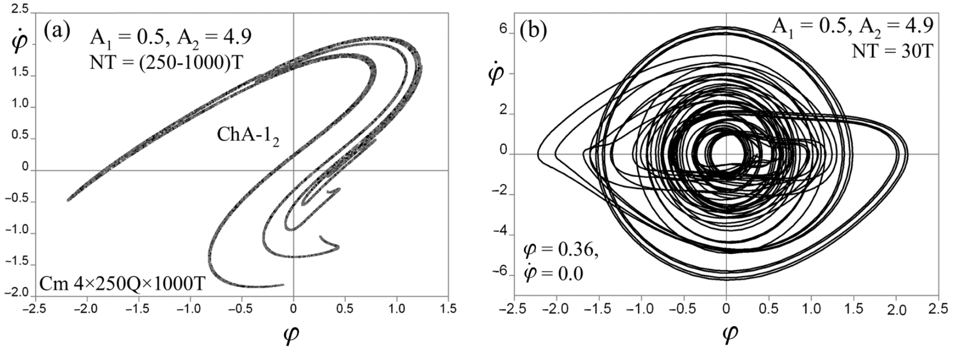

Fig. 3. Chaotic attractor in the parametrically excited pendulum system for cross-section $A_2 = 4.9$ of the bifurcation diagram in Fig. 2: (a) Poincaré map – Cm 4 × 250Q × (250–1000)T; (b) phase projection with NT = 30T.

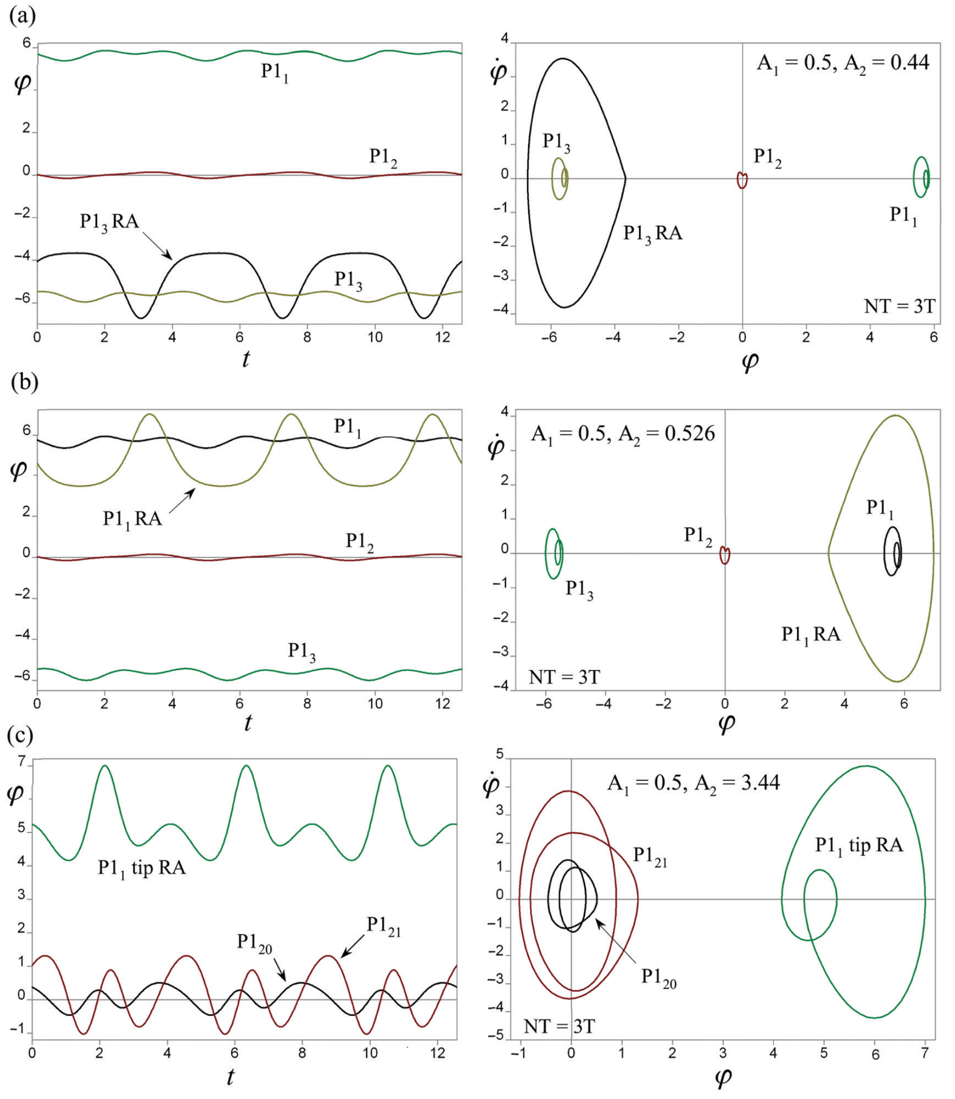

Fig. 4. Coexistence of P1 usual and rare attractors P1 RA for three cross-sections (see Fig. 2): (a) time histories and phase projections for $A_2 = 0.44$, the rare attractor $P1_3$ RA has the fixed point FP (–4.05606/1.17632); (b) the same for cross-section $A_2 = 0.526$, the rare attractor $P1_1$ RA has the FP (4.56968/–2.60245); (c) the same for $A_2 = 3.44$, the rare attractor $P1_1$ tip RA has the FP (5.23616/–0.315143).

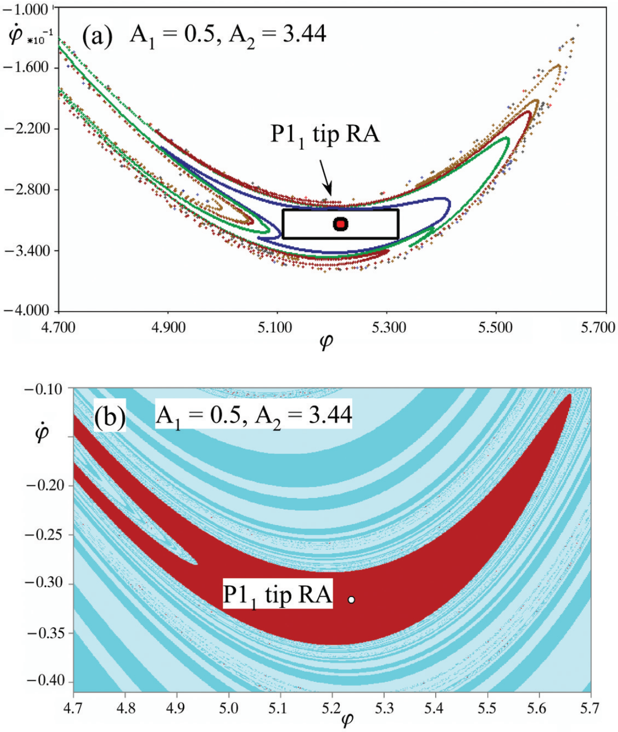

Fig. 5. Domains of attraction of period 1 tip rare attractor $P1_1$ tip RA (see Fig. 2a) in a parametrically excited pendulum system: (a) obtained by using a reverse contour mapping from a rectangle; (b) obtained by using cell-to-cell mapping with 501 × 501 grid of initial conditions.

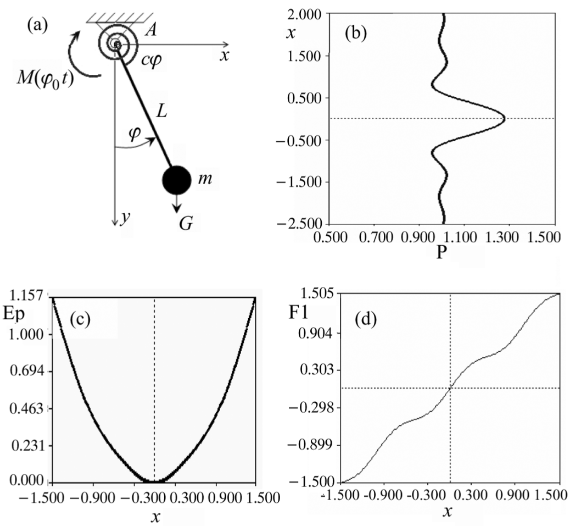

Fig. 6. The driven damped pendulum system with a linear restoring moment and with the external periodic excited moment. (a) Physical model; (b) backbone curve; (c) potential well; (d) restoring moment.

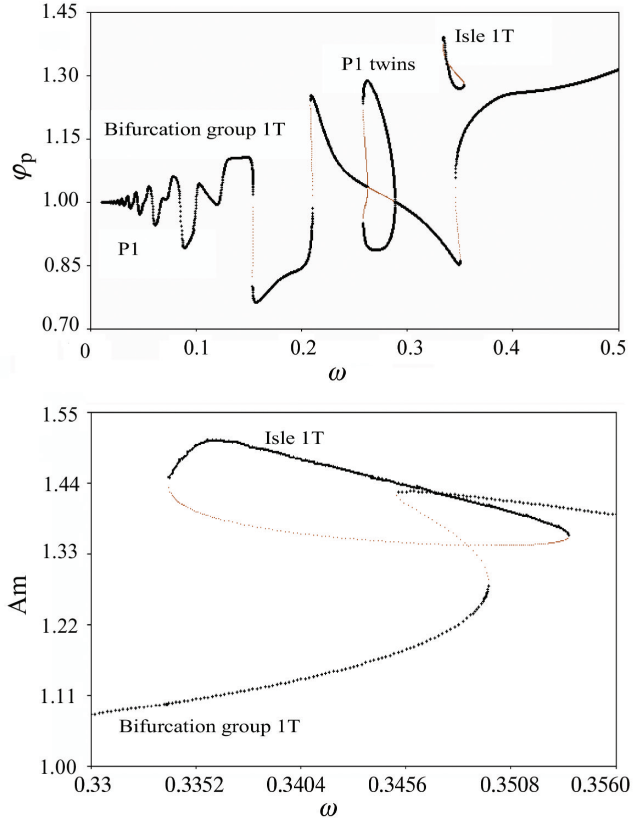

Fig. 7. The driven damped pendulum system (Eq. (2)) with a linear restoring moment and with the external periodic excited moment. The system has two 1T bifurcation groups: usual P1 and isle P1.

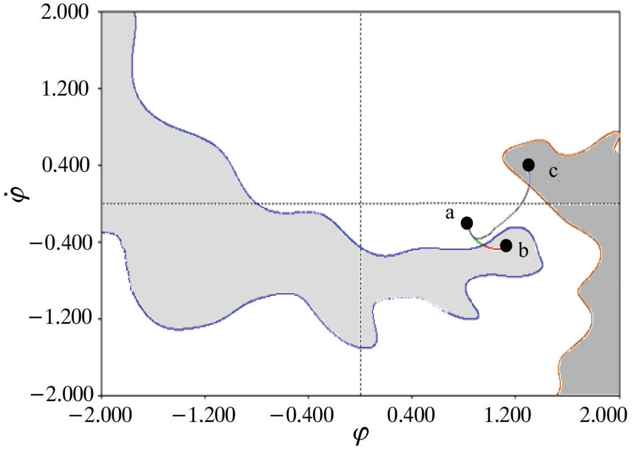

Fig. 8. Domains of attraction of three attractors P1, $\varphi = 0.347$.

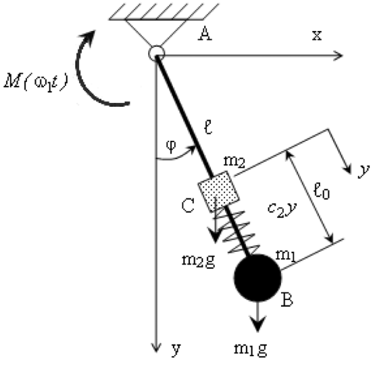

Fig. 9. The driven damped pendulum model with a sliding mass and with the external periodic excited moment.

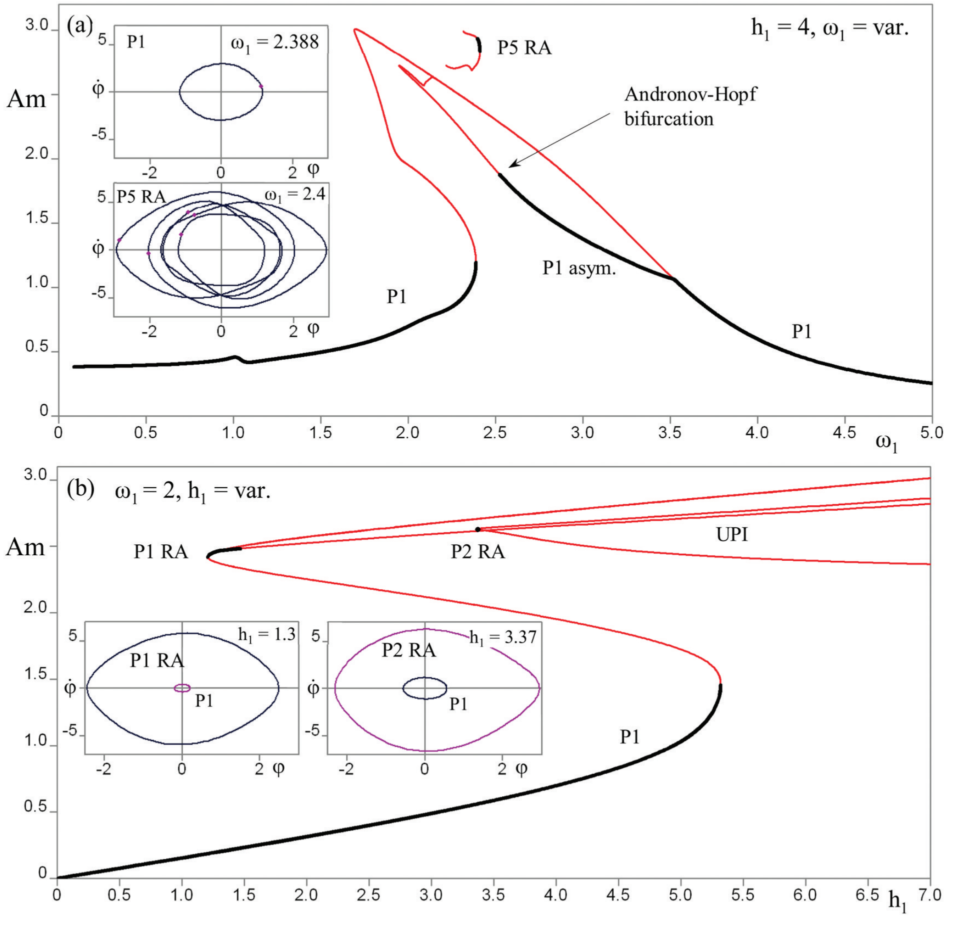

Fig. 10. The driven damped pendulum system with a sliding mass and with the external periodic excited moment. Bifurcation diagrams – amplitude of the pendulum Am vs frequency ω_1 and amplitude h_1 of excitation: (a) $h_1 = 4$, ω_1 = var.; (b) $\omega_1 = 2$, h_1 = var. The pendulum system has several bifurcation groups with their own tip type RAs.

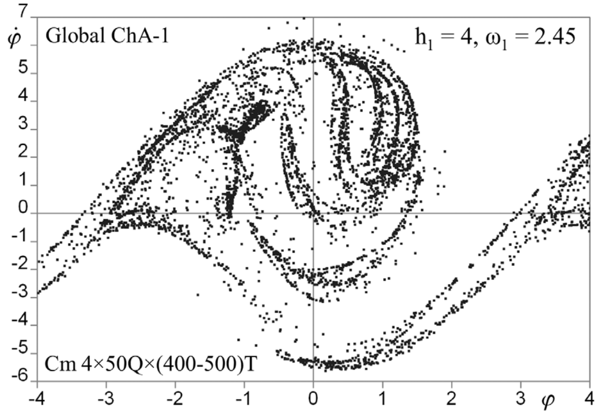

Fig. 11. Global chaos ChA-1 on the Poincaré map Cm 4 × 50Q × (400–500)T in the pendulum system (Eq. (3)) with a sliding mass for cross-section of the bifurcation diagram (Fig. 10).

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Juhitava sumbumisega pendelsüsteemide täielik bifurkatsioonianalüüs

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Tehnikas kasutatakse laialdaselt pendelsüsteeme, kuid nende kvalitatiivset käitumist pole piisavalt uuritud. Antud töö eesmärgiks oli uurida kolme juhitava sumbumisega pendelsüsteemi uusi mittelineaarseid toimemehhanisme, mis on piisavalt lähedased masinate ja mehhanismide dünaamikas kasutatavatele reaalsetele mudelitele. Artiklis on analüüsitud juhitava sumbumisega pendelsüsteemide uute bifurkatsioonigruppide olemasolu, haruldasi atraktoreid ja kaootilisi režiime.