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On wave propagation in laminates with two substructures

Mihhail Berezovski, Arkadi Berezovski, Tarmo Soomere and Bert Viikmäe

Centre for Nonlinear Studies, Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia; misha@cens.ioc.ee

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Abstract. We study numerically the influence of the presence of a complex internal structure of laminates, consisting of layers of different properties and variable thickness, on the dynamic response of the material. The influence of the internal structure of laminate layers on the signal propagation is demonstrated by several examples for periodic and double periodic laminates. It is also discovered that the influence of the mutual position of layers with different internal structure can be significant.

Key words: wave propagation, multistructured solids, numerical simulations, laminates.

1. INTRODUCTION

The behaviour of many materials used in engineering (e.g., metals, alloys, granular materials, composites, liquid crystals, polycrystals) is often influenced by the existing or emergent microstructure (e.g., phases in multiphase materials, voids, microcracks, dislocation substructures, texture). In general, the components of such a microstructure have different material properties, resulting in a macroscopic material behaviour like in highly anisotropic and inhomogeneous materials.

Due to the complex structure of such media, wave propagation is accompanied by reflection, refraction, diffraction and scattering phenomena that are difficult to quantify [1]. Small-scale changes in a heterogeneous material's microstructure can have major effects in its macro-scale behaviour. For example, alloying elements, nano-reinforcements and the crystalline structures of polymers all have profound effects on the parental material's macroscopic response [2]. Modelling macroscopic mechanical properties of materials in relation to their substructure is a long-standing problem in materials science [3].

The development of new materials as well as the optimization of classical materials requires modelling, more closely related to the substructure of the materials under consideration. The diversity of possible situations, as far as the geometry, the scale and the contrast of multiphase structures are concerned, is huge. In dynamic problems, the role of the scale effects is significant. When the wavelength of a travelling signal is comparable with the characteristic size of heterogeneities, successive reflections and refractions of the local waves at the interfaces lead to the dispersion and attenuation of the global wave field. Besides the geometrical characteristics of multiphase materials, an important aspect is the contrast between different constituent materials.

In order to model the mechanical behaviour of such a variety of heterogeneous materials, the substructure has to be simplified. As a first approximation, much useful information can be inferred from the analysis of wave propagation in a body where the periodic arrangement of layers of different materials is confined within a finite spatial domain [3].

In the framework of the general study of wave propagation in solids with microstructure [4], the influence of multiple reflections at internal interfaces on wave propagation in layered composites of two different materials was investigated numerically [5], and the corresponding model of microstructure was validated [6].

Usually, real materials contain more than one substructure. It is heuristically obvious that each substructure gives its own contribution to the total material behaviour. In order to construct an appropriate model of response of a material with more refined internal structures, the first step is understanding the behaviour of the material with at least two different substructures.

The aim of this paper is to investigate the influence of the presence of a more complex internal structure of laminates on the dynamic response of the material. For this purpose, we use numerical simulations of one-dimensional wave propagation in materials with several compositions of two substructures.

The modification of the wave-propagation algorithm introduced in [7] is applied as a basic tool of numerical simulations due to its physical soundness, accuracy and thermodynamic consistency [8].

2. GOVERNING EQUATIONS

The simplest example of heterogeneous media is a periodic laminate composed of materials with different properties. One-dimensional wave propagation in the framework of linear elasticity is governed by the conservation of linear momentum [9]

$$\rho(x)\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0, \tag{1}$$

and the kinematic compatibility condition

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x}. \tag{2}$$

Here t is time, x is space variable, the particle velocity $v = u_t$ is the time derivative of the displacement u, the one-dimensional strain $\varepsilon = u_x$ is the space derivative of the displacement, σ is the Cauchy stress and ρ is the material density. The compatibility condition (2) follows immediately from the definitions of the strain and the particle velocity.

Equations (1) and (2) contain three unknowns: v, σ and ε. The closure of this system is achieved by a constitutive relation, which in the simplest case is the Hooke's law

$$\sigma = \rho(x)c^2(x)\,\varepsilon, \tag{3}$$

where $c(x)$ is the corresponding longitudinal wave velocity. The indicated explicit dependence on the location in space x means that the medium is materially inhomogeneous. The resulting system of equations (1)–(3) is solved numerically.

3. NUMERICAL SIMULATION

The cross-differentiation of Eqs. (1) and (2), after Hook's law (Eq. (3)) has been applied, leads to the classical variable-coefficient wave equation. The solution of this wave equation is well-known if the coefficients, characterizing spatial variations of the background environment (here interpreted as the properties of the material), are smooth. These variations and thus the coefficients of the wave equation obviously are not smooth near discontinuities in the material parameters. Therefore, one cannot employ standard methods for solving such equations that often fail completely if the parameters vary drastically on the grid size.

By contrast, the recently developed high-resolution wave-propagation algorithm [10] has been found well suited for the modelling of wave propagation in rapidly-varying heterogeneous media [11]. Within this algorithm, every discontinuity in parameters is accounted for by solving the Riemann problem at each interface between discrete elements. As a result, the reflection and transmission of waves at each interface are handled automatically for a wide range of parameters of the inhomogeneities.

The general idea of methods with such abilities is the following. The division of a body into a finite number of computational cells is accompanied by the description of all fields inside the cells as well as by accounting for the interaction between neighbouring cells. By doing so it is possible to approximate the required fields inside the cells in a smooth manner and simultaneously describe the discontinuities of the fields at the boundaries between cells. This way of interaction leads to the appearance of certain excess quantities, which represent the difference between the exact and approximate values of the fields. The interactions between neighbouring cells are described by means of fluxes at the boundaries of the cells. These fluxes correspond to the excess quantities, which can be calculated by means of jump relations at the boundaries between cells.

High-resolution finite-volume methods, capable of handling discontinuities in such a manner, were originally developed for capturing shock waves in solutions to non-linear systems of conservation laws, such as the Euler equations of gas dynamics [7]. However, they are also well suited to solving non-linear wave propagation problems in heterogeneous media containing many sharp interfaces. Recently, a wave-propagation algorithm of this type was successfully applied to one-dimensional non-linear elastic waves in a heterogeneous periodic medium consisting of alternating thin layers of two different materials [12,13].

An improved composite wave propagation scheme, where a Godunov step is performed after several second-order Lax–Wendroff steps, was successfully applied for the two-dimensional thermoelastic wave propagation in media with rapidly varying properties [14–16].

4. RESULTS OF NUMERICAL SIMULATION

To investigate the influence of two substructures on the material behaviour, the propagation of a pulse in a one-dimensional medium, which can be interpreted as an elastic bar, is considered. This bar is assumed homogeneous except of a region of length l in the middle of the bar, which contains periodically alternating layers of thickness d (Fig. 1). Total length of the bar L is equal to $5000\Delta x$. We set the length l of the inhomogeneous domain equal to $900\Delta x$ for all numerical simulations (Δx is the constant space step used in simulations).

The density and the longitudinal velocity in the bar are chosen as $\rho = 4510\,\mathrm{kg/m}^3$ and $c = 5240$ m/s, respectively. The shape of the pulse is formed by an excitation of the strain at the left boundary for a limited time period $(0 < t < \lambda\Delta t)$

$$u_x(0,t) = (1 + \cos(2\pi(t - \frac{\lambda}{2}\Delta t)/\lambda), \tag{4}$$

where λ is the pulse length. After that the strain is equal to zero. Consistency condition for velocity is also used. At the right boundary we apply non-reflective boundary conditions. The arrow at the left end of the bar in Fig. 1 shows the direction of the pulse propagation. Numerical simulations were performed for

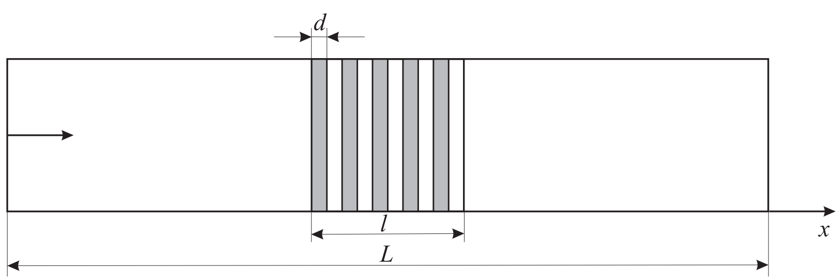

Fig. 1. Geometry of the problem.

three different lengths of a pulse: $\lambda = 30\Delta x$, $90\Delta x$ and $180\Delta x$ for each analysed substructure composition.

We consider several substructure compositions within the inhomogeneous domain (Fig. 2). We assume that the material of the bar itself is the hardest and of the highest density, and will call it "hard" material in what follows. The substructure may contain two different materials. The one with the lowest density and longitudinal velocity ($\rho = 2703\ \mathrm{kg/m^3}$ and $c = 5020$ m/s, respectively) we call "soft" and the one with the intermediate parameters ($\rho = 3603\ \mathrm{kg/m^3}$ and $c = 5100$ m/s) we call "intermediate" material. The smallest scale of the substructure – the minimum size of each sublayer – is set equal to $30\Delta x$.

The results of propagation of the pulses through different compositions of the substructure are compared with the behaviour of the reference pulses, the propagation of which is calculated for a homogeneous bar of the "hard" material. The resulting pulse is recorded at 4600 time steps.

We start the analysis from a simple periodic composition of two materials ("hard" and "soft") with a fixed size of layers $d = 90\Delta x$. This composition is represented as case (a) in Fig. 2. The results of numerical simulations of the pulse propagation (Fig. 3) demonstrate that the shape of the resulting pulse is considerably modified depending on the length of the initial pulse. The stress is normalized by the amplitude of the initial pulse. The initial pulse is separated into two leading pulses. The tail of the signal contains a negative disturbance. These results are qualitatively similar to earlier studies of pulse propagation in similar layered medium [5].

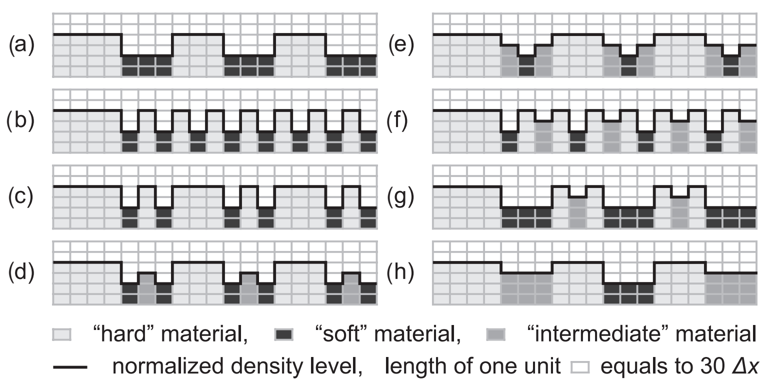

Fig. 2. Substructure compositions. The bold line describes the relative density of the material. The light grey layers represent the material of the bar ($\rho = 4510\ \mathrm{kg/m^3}$, $c = 5240$ m/s), shaded layers indicate the "intermediate" substructure material ($\rho = 3603\ \mathrm{kg/m^3}$, $c = 5100$ m/s) and the dark grey layers – the "soft" substructure material ($\rho = 2703\ \mathrm{kg/m^3}$ and $c = 5020$ m/s).

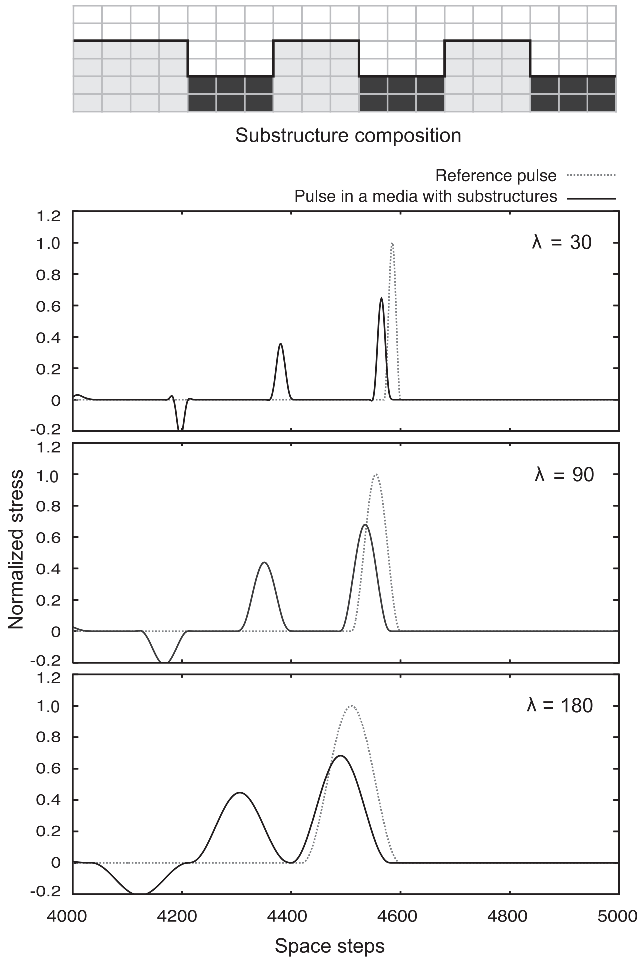

Fig. 3. Pulse shape at 4600 time steps for "hard–soft" structure laminate ($d = 90\Delta x$, case (a) in Fig. 2).

A decrease in the size of periodically alternating "hard" and "soft" layers d to $30\Delta x$ under otherwise identical conditions of the properties of the medium, pulses and the numerical simulations (case (b) in Fig. 2) leads to a considerable response of the pulses to the substructure (Fig. 4). The propagation of the longest pulse ($\lambda = 180\Delta x$) shows the lowest rate of distortions: the initial positive pulse almost keeps its shape but relatively small and smooth disturbances are created. For the intermediate length of the pulse ($\lambda = 90\Delta x$) quite a complex transition of the pulse into a multi-peaked positive signal, consisting of a sequence of several overlapping positive pulses, takes place. This process is accompanied by excitation of an irregular but again mostly smooth tail. The propagation of the shortest pulse ($\lambda = 30\Delta x$) over this substructure leads to the formation of two clearly separated leading pulses. In contrast to the previous case, the first leading pulse is now

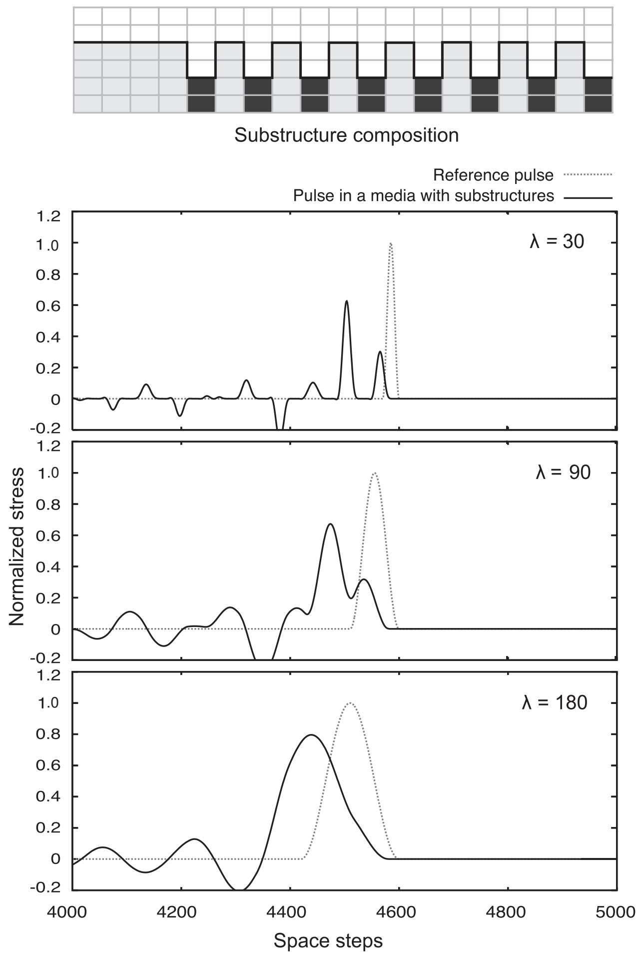

Fig. 4. Pulse shape at 4600 time steps for “hard–soft” structure laminate ($d = 30\Delta x$, case (b) in Fig. 2).

smaller than the second one. These results show that the dispersion is stronger if the wavelength is comparable with the size of the inhomogeneity.

Case (c) in Fig. 2 can be interpreted as a combination of the previous two cases. The bar contains here a more complex substructure consisting of a succession of regions that involve two thin layers of soft material separated by a similar layer of hard material ($d = 30\Delta x$ for each thin layer as in case (b)) alternating with regions equivalent to a thicker layer ($d = 90\Delta x$) of the “hard” bar material as in case (a). Figure 5 shows that the longest pulse ($\lambda = 180\Delta x$) is clearly less distorted than in the case of substructure with very thin layers whereas the distortions are more pronounced than in the case with thick internally homogeneous sublayers. The structure of the leading pulse basically survives the interaction with the substructure. Shorter pulses, however, show separation into two leading pulses of almost equal length and height. This example shows that even relatively

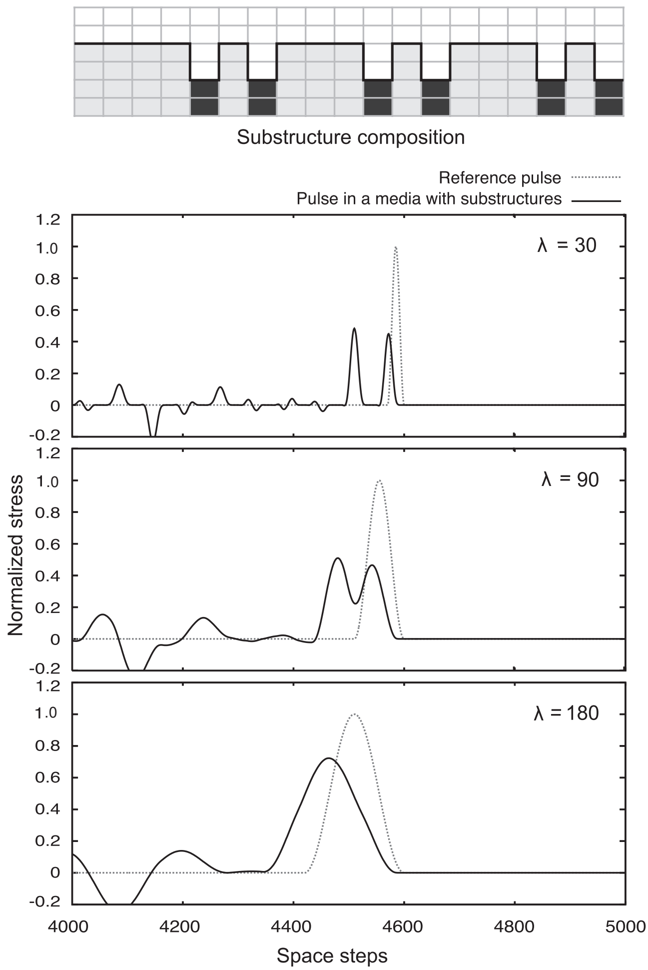

Fig. 5. Pulse shape at 4600 time steps for mixed "hard–soft" structure laminate ($d = 30\Delta x$ and $90\Delta x$, case (c) in Fig. 2).

small changes in the substructure (in this case equivalent to shifting the layers with internal fine structure to some distance from each other) leads to quite significant effects in pulse propagation.

The reaction of the signal to the presence of substructure is even more complicated if the regions containing three thin layers of different material are formed from materials of different properties.

The simplest way to introduce such a second substructure is to replace the thin layer of "hard" material between the layers of "soft" material by an equally thin layer of "intermediate" material. The composition of the inhomogeneous domain then corresponds to case (d) in Fig. 2. The influence of the second substructure on the pulse propagation (Fig. 6) is evident, compared, for example, with the reaction of the pulse propagation to change in the material composition from

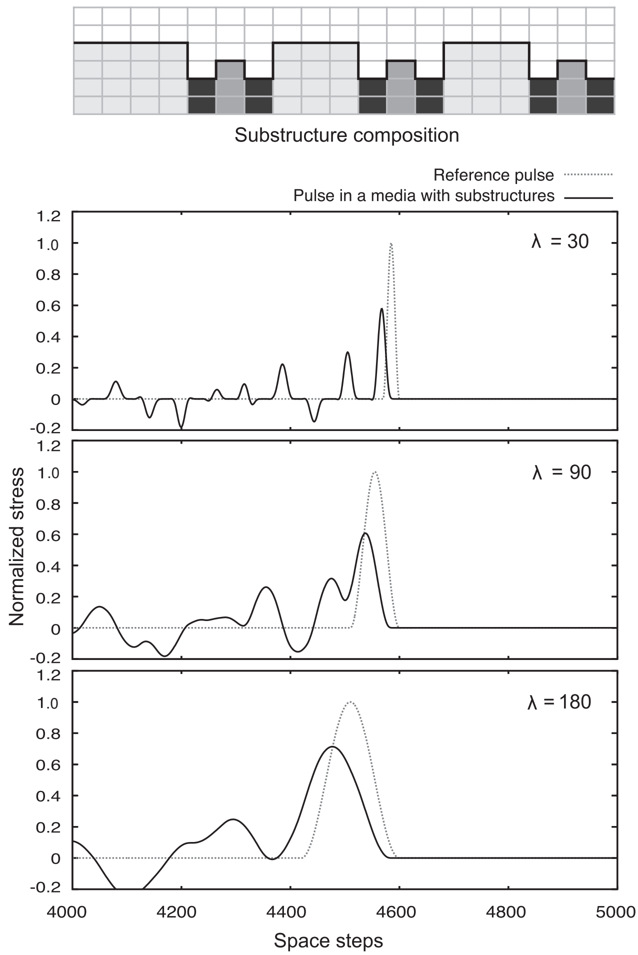

Fig. 6. Pulse shape at 4600 time steps for "hard–soft–intermediate–soft" double structure laminate ($d = 30\Delta x$ and $90\Delta x$, case (d) in Fig. 2).

case (b) to case (c). Only the final shape of the longest pulse is similar to that of the previous case. For the shorter pulses, the first pulse of the two leading ones is now higher than the second one in contrast to the previous case. Quite large deviations from the above cases become evident for the pulse with a duration of $180\Delta x$ that contains now strong oscillations in the tail. Therefore, the introduction of the second substructure even in quite a limited manner leads to clearly observable changes in the dynamic response of the pulses to the structure of the laminate.

Completely different results are obtained for the inverse order of "soft" and "intermediate" substructure layers (case (e) in Fig. 2). Now we have the periodic composition of two thin layers of "intermediate" and one thin layer "soft" material ($d = 30\Delta x$) alternating by thick layers of "hard" material ($d = 90\Delta x$). For all pulse lengths ($\lambda = 30\Delta x$, $90\Delta x$ and $180\Delta x$) we observe only one leading pulse

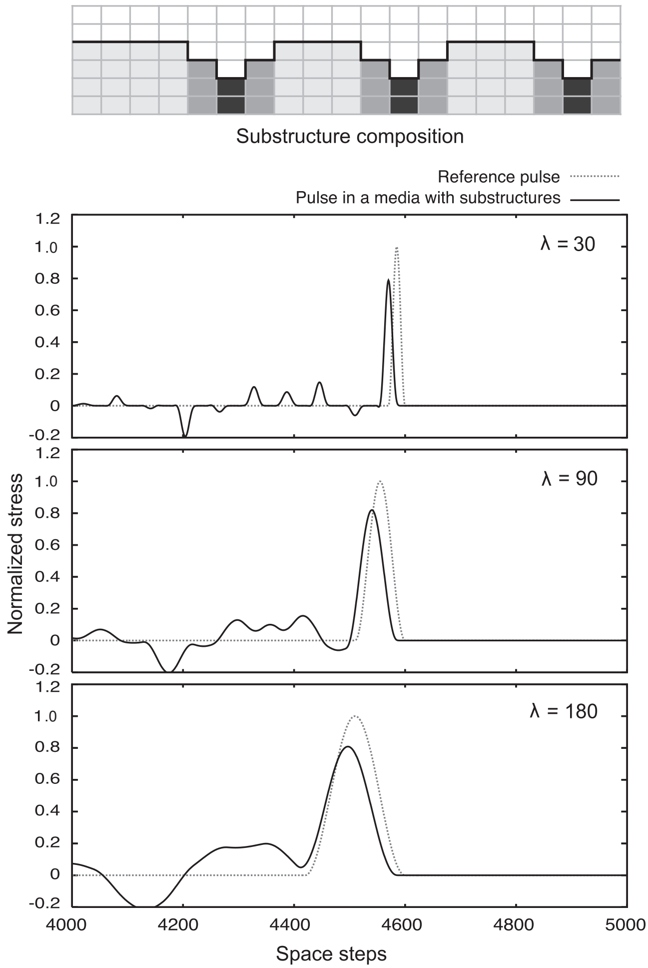

Fig. 7. Pulse shape at 4600 time steps for “hard–intermediate–soft–intermediate” double structure laminate ($d = 30\Delta x$ and $90\Delta x$, case (e) in Fig. 2).

(Fig. 7). Interestingly, the distortions of the signal propagation are relatively small and the initial shape of the pulse is clearly evident. As a consequence, the leading pulse is the highest one for all lengths of the initial pulse. This feature demonstrates that not only the presence of the second-level substructure influences the material behaviour, but also the relative position of both substructures is significant.

In our next composition of the inhomogeneous domain we set the periodic alternating thin layers of two substructures of “soft” and “intermediate” materials with one thin layer of “hard” material $d = 30\Delta x$. This composition corresponds to the case (f) in Fig. 2. The results of numerical simulations of the pulse propagation for this composition (Fig. 8) are very similar to the case of the simple periodic “hard–soft” substructure with $d = 30\Delta x$ shown in Fig. 4. The influence of the second substructure can be recognized, as expected, by somewhat smaller

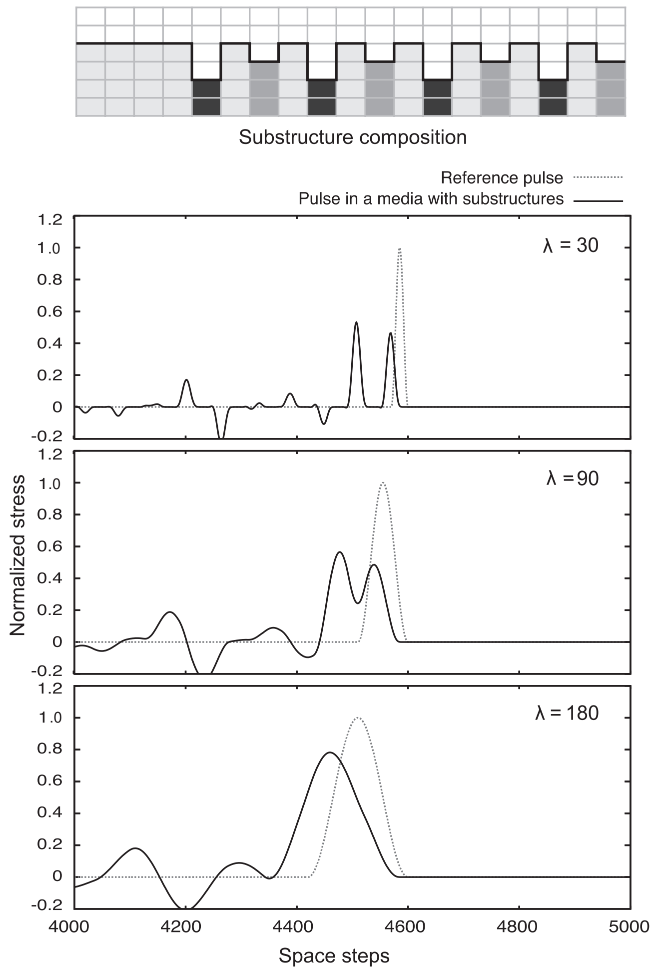

Fig. 8. Pulse shape at 4600 time steps for "hard–soft–hard–intermediate" double structure laminate ($d = 30\Delta x$, case (f) in Fig. 2).

distortions of the initial pulse for all three simulations.

In a variation of case (d) in Fig. 2, we replace the position of the "intermediate" material thin layer as in case (g). We set it in the centre of a thick "hard" material layer instead of the "soft" material layer. Here we alternate the thick layer of "soft" material ($d = 90\Delta x$) with a combination of two thin layers of "hard" material with one thin layer of "intermediate" material ($d = 30\Delta x$). Corresponding results of numerical simulations (Fig. 9) are similar to those in Fig. 6, only the amplitudes of distortions of resulting pulses are slightly changed.

The last composition considered in this paper repeats the very first composition of simple thick periodic "hard–soft" composition (Fig. 2, case (a)), but every second "soft" layer is replaced by the "intermediate" material (case (h)). In this composition the thick "hard" material layers are alternating with thick

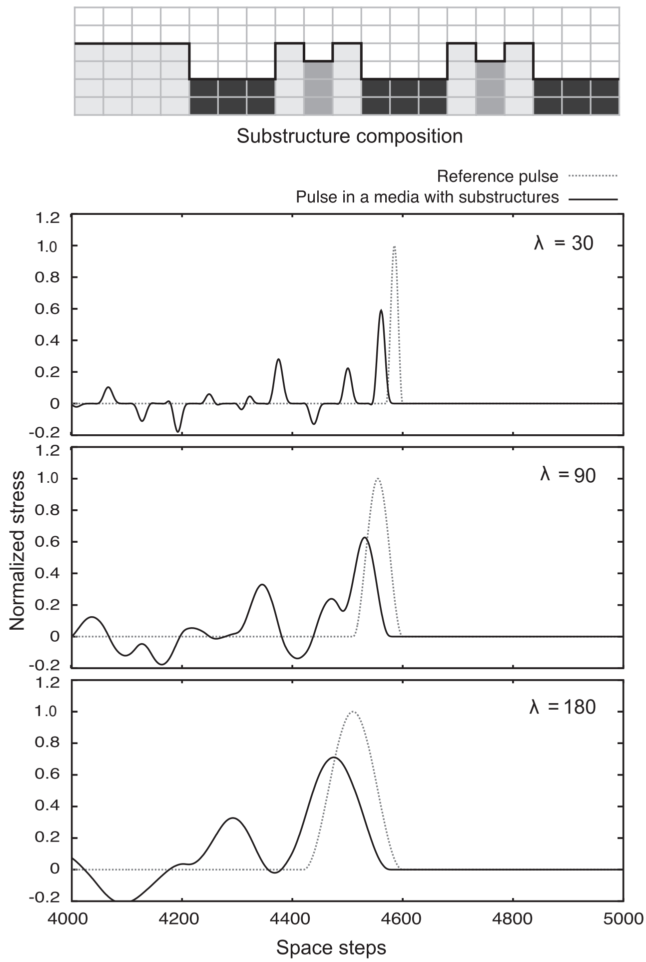

Fig. 9. Pulse shape at 4600 time steps for "hard–soft–hard–intermediate–hard" double structure laminate ($d = 30\Delta x$ and $90\Delta x$, case (g) in Fig. 2).

"intermediate" material layers and thick "soft" layers ($d = 90\Delta x$). The observed results (Fig. 10) are very similar to those obtained for the first case with only one substructure (Fig. 4). The influence of the second substructure in terms of the presence of layers with a smaller difference of their physical properties from the bar material becomes evident as somewhat better match of the final shape of the pulses with the reference pulses.

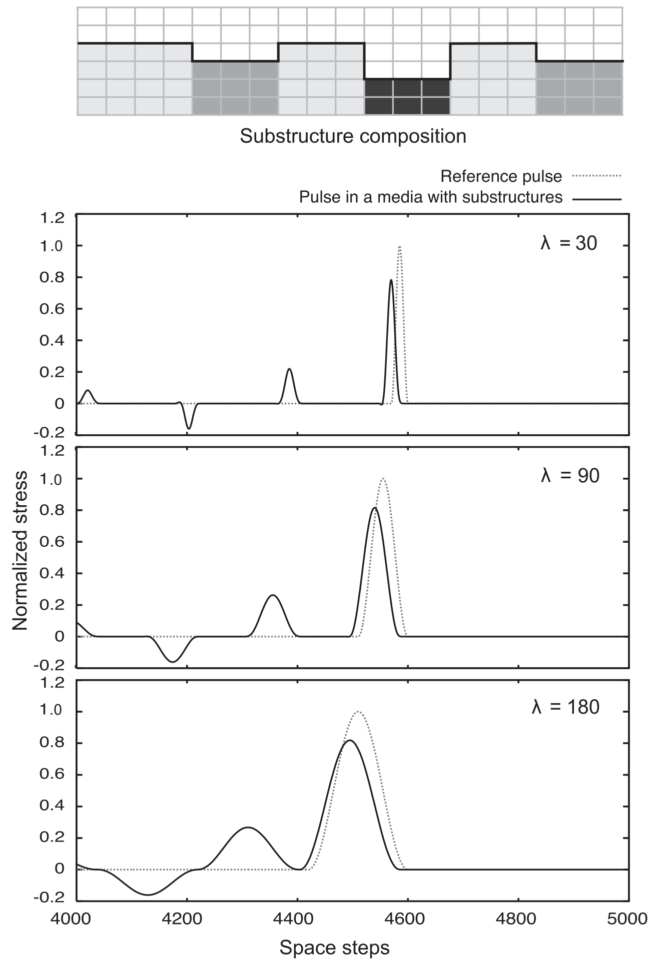

Fig. 10. Pulse shape at 4600 time steps for "hard–intermediate–hard–soft" double structure laminate ($d = 90\Delta x$, case (h) in Fig. 2).

5. CONCLUSIONS

It is not unexpected that the introduction of a more complex structure to laminate materials (called second substructure here) affects the dynamic response of the signal propagation through the laminate. The presented results of numerical simulations confirm the importance of a second substructure to the behaviour of pulses. A significant outcome from the presented numerical simulations is that not only the presence of the second substructure, but also its internal geometry and the mutual distribution of the hard and soft layers is significant. This influence of the second substructure should be taken into account by further developments in the modelling of the dynamic response of microstructured solids.

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Lainelevi keeruka stuktuuriga laminaatides

Mihhail Berezovski, Arkadi Berezovski, Tarmo Soomere ja Bert Viikmäe

On analüüsitud lainelevi omadusi keeruka struktuuriga laminaatides, kus erinevatel kihtidel võib olla erinev paksus ja sisemine struktuur, kasutades selleks positiivseid impulsse. Kihtide sisestruktuuri mõju signaali levikule on demonstreeritud mitmetes erineva paksuse ja sisemiste kihtide asetusega materjalides. On näidatud, et signaali mõjutab märgatavalt isegi erinevate omadustega kihtide omavaheline järjestus.