Rigid body impact models
partially considering deformation

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Abstract. The paper deals with the low-speed impact of rigid bodies. The elasticity and damping forces, acting on a body during the impact, depend on the physical properties of the material; in this work these forces are considered according to the Kelvin–Voigt, Bingham and Maxwell models of the medium. New models of the forces, acting during the impact, are elaborated. Numerical solutions of the equations for various impact forces are obtained.

Key words: impact, impact forces, duration of impact, impact deformations.

1. INTRODUCTION

Collisions of moving bodies are inherent on all levels of Universe – from microcosm up to space; therefore the shock phenomena are rather diverse. Specific features of impacts are their rapidity and intensity. They can appear both useful, as for example the piles driving, vibro-impact systems, extraction of ore or in ball games or harmful, as for example transport accidents. Hence, the problem of impact is very important not only for theorists, but also for designers, motorcar drivers, sportsmen, etc. Classification of impacts depends on the distribution of the deformations in each of the colliding bodies and how these deformations influence the period of contact. Local deformations during the impact depend on the initial velocity of the contact point and on the rigidity of the bodies [1–3]. In this work the low-speed impact that does not cause inter-penetration of the bodies is considered. The low-speed impact leads to the contact pressure, which gives small deformations that are significant only in the

small area adjoining the contact area. Impulsive forces are a response to the deformations, arising near the contact area and propagating in the bodies. Mathematical models reflect this process more or less completely.

In the classical theory of rigid body impact, deformations are not taken into account and only integrated characteristics of the impact forces (their impulses) are considered. This approach involves the application of fundamental laws of mechanics to predict the velocity after impact. Impulse-momentum equation $\overline{S} = m\Delta\overline{v}$ (m is mass of the body and $\overline{v}$ is velocity), connecting pre-impact and post-impact velocity, forms the core of this approach. Algebraic nature of this method makes the mathematical development easy and accessible to most engineers, but this approach is unable to predict the contact force between the bodies or stress in them [4].

When the body is completely deformed during the impact, the wave theory, which describes the stress state of bodies most completely, is applied. This theory is based on complex equations, having an exact solution only in specific cases.

The models, partially considering deformations, are intermediate between these two extreme approaches. In these models it is supposed that motion of all the body during impact is described by the differential equations of motion of a rigid body $\overline{F} = m\overline{a}$ ($\overline{F}$ is force and $\overline{a}$ is acceleration) [5]. The forces, acting during the impact time take into account viscoelastic properties of real bodies and are modelled by a set of springs and dampers [4,6,7]. Such models are discussed in this paper; both linear and non-linear models of the elastic force and damping forces are considered. Our aim is to find the most suitable model of frictionless impact for the calculation of vibro-impact systems, in particular, for the determination of the motions, accompanied by impacts.

2. MODELS OF IMPACT FORCES FOR THE KELVIN–VOIGT MEDIA

Kelvin–Voigt media involve the parallel-connected linear spring and damper and the mechanism of impact contact is modelled by the second order linear differential equation

$$m\ddot{x} = -cx - b\dot{x}, \tag{1}$$

where c is the spring rigidity factor and b is the damping factor.

This model permits analytical definition of the law of motion $x(t)$, the change of the velocity $v(t)$ and impulsive force $F(t)$ during the impact [8]:

$$x(t) = \frac{v_0}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t}), \tag{2}$$

$$v(t) = \frac{v_0}{\lambda_1 - \lambda_2}(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}), \tag{3}$$

$$N(t) = \frac{v_0}{\lambda_1 - \lambda_2}(\lambda_1^2 e^{\lambda_1 t} - \lambda_2^2 e^{\lambda_2 t}). \tag{4}$$

In Eqs. (2)–(4) v_0 is the pre-impact velocity, λ_1 and λ_2 are roots of the characteristic equation for Eq. (1):

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4mc}}{2m}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4mc}}{2m}.$$

The time moment, when the impact force $N(t)$ becomes equal to zero, determines the duration of the impact T. For the duration of impact to have a real value, the rigidity of the spring must be large; then λ_1 and λ_2 are complex numbers and the duration of impact T will be

$$T = \frac{4m \arctan\left(\dfrac{\sqrt{4mc - b^2}}{b}\right)}{\sqrt{4mc - b^2}}. \tag{5}$$

In this case the coefficient of restitution of the velocity e does not depend on the pre-impact velocity and is determined by

$$e = \exp\left(\frac{-\pi b}{\sqrt{4mc - b^2}}\right).$$

In Fig. 1 the plots of $T(b,c)$ and $e(b,c)$ are presented for a body with a mass of $m = 1\,\text{kg}$, where the rigidity c changes from 2×10^6 to $2 \times 10^7\,\text{N/m}$ and b from 200 to 1000 N/m/s. It is obvious that the duration of impact is mostly influenced by the elastic part of the impact force.
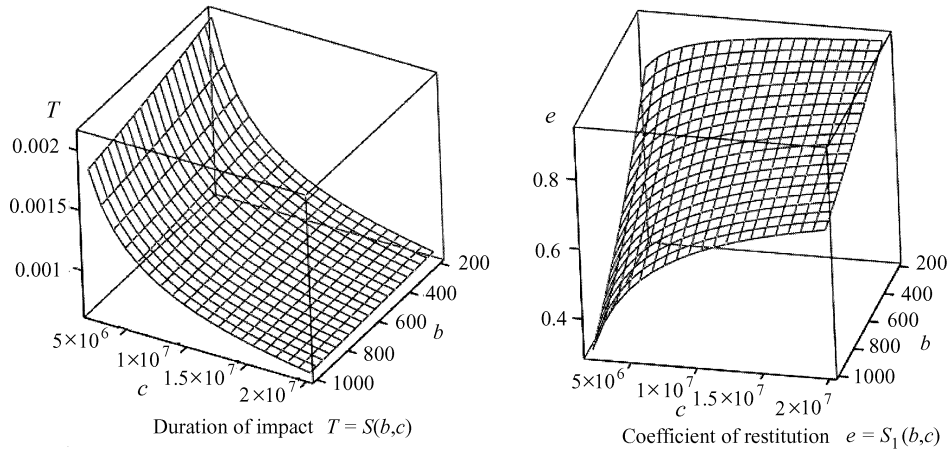
In Fig. 2 the dependence of the impact force N and its elastic part cx on the displacement and the dependence of the velocity v on the displacement in case of the Kelvin–Voigt model are presented for $m = 1\,\text{kg}$, $c = 2 \times 10^6\,\text{N/m}$, $v_0 = 2.5\,\text{m/s}$, $b = 400\,\text{N/m/s}$; $e = 0.69$.

The disadvantage of this model is the occurrence of tensile forces by the end of the impact that is physically impossible.

For the elimination of this disadvantage, some authors suggest to multiply the damping part of the impact force by displacement; then the equation of motion during impact will be

$$m\ddot{x} = -cx - b_1 x \dot{x}. \tag{6}$$

Here the value of the damping factor b_1 is different from the value of b in Eq. (1). By identical duration of the impact in both cases, the coefficients of restitution e will be different.

Fig. 1. Dependence of the duration of impact T and of the coefficient of restitution e on the coefficients b and c in the case of Kelvin–Voigt model.
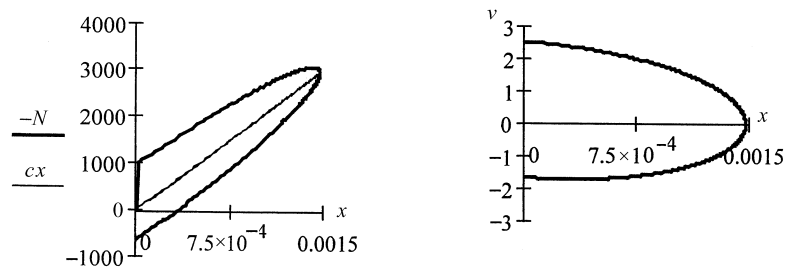
Fig. 2. Dependence of the impact force N and its elastic part cx on the displacement and dependence of the velocity v on the displacement in case of the Kelvin–Voigt model ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b = 400$).

In Fig. 3 the dependence of the impact force N and velocity v on the displacement in accordance with Eq. (6) for $m = 1\,\text{kg}$, $c = 2 \times 10^6\,\text{N/m}$, $v_0 = 2.5$ m/s, $b = 4$ N/m/s, are shown; in this case $e = 0.78$.
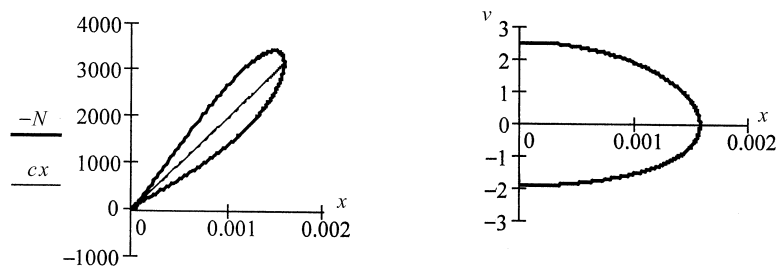
Fig. 3. Dependence of the impact force N, its elastic part cx and velocity v on the displacement in the case of the changed damping part in the Kelvin–Voigt model ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b_1 = 4 \times 10^5$).

The disadvantage of this model is that the variation of b_1 influences the final velocity of impact only a little; thus the coefficient of restitution e becomes large.

3. MODELS OF IMPACT FORCES FOR THE BINGHAM MEDIA

The model of the Bingham media differs from the Kelvin–Voigt model by an additional force P (an analogue to dry friction), the direction of which depends on the direction of the velocity. In this case the equation of motion of a body during impact is

$$m\ddot{x} = -cx - b\dot{x} - P\operatorname{sgn}(v). \tag{7}$$

The force P does not influence the time of impact, but it reduces the coefficient of the restitution of velocity in comparison with the Kelvin–Voigt model. An example is shown in Fig. 4.

In this model a tensile force is also generated; it is impossible to eliminate this force by multiplying the damping part of the force by displacement. Such a motion is described by the equation

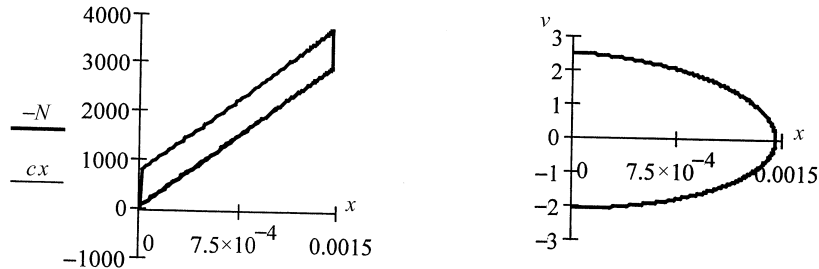$$m\ddot{x} = -cx - b_1 x\dot{x} - P\operatorname{sgn}(v). \tag{8}$$

An example is shown in Fig. 5.
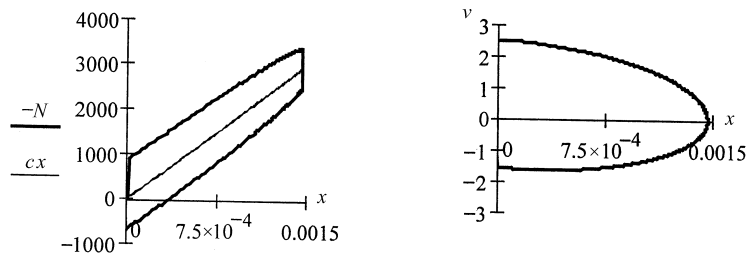
Fig. 4. Dependence of the impact force N, its elastic part cx and velocity v on the displacement in case of the Bingham media ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b = 200$, $P = 400$); $e = 0.66$.
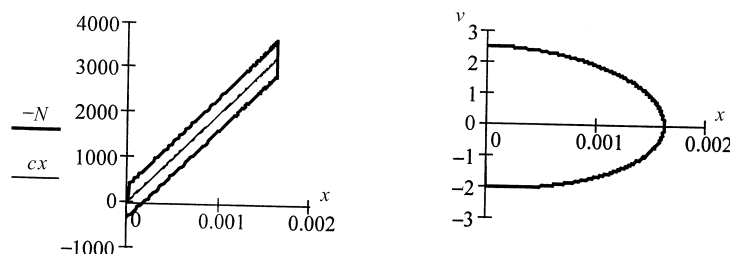
Fig. 5. Dependence of the impact force N, its elastic part cx and velocity v on the displacement in case of the changed Bingham media ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b = 200$, $P = 400$); $e = 0.79$.

Fig. 6. Dependence of the impact force N, its elastic part cx and velocity v on the displacement in the case of the Bingham medium without the damping part ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $P = 800$); $e = 0.80$.

If in the Bingham model we do not take into account the damping force and take into account the constant force P in one direction – in compression phase, the tensile stress does not appear. The differential equation that describes the motion during impact is in this case

$$m\ddot{x} = -cx - P \, if \, (v \leq 0, 0, 1). \tag{9}$$

The Mathcad logical function if(cond, a, b) returns a if logical condition is true, b otherwise.

Dependence of such a force on the displacement is presented in Fig. 6.

The independence of the damping from the velocity is a disadvantage of this model.

4. THE MODEL OF THE IMPACT FORCE FOR THE MAXWELL MEDIUM

The model of the viscoelastic Maxwell medium consists of a spring and a damper, connected consistently, thus the force P and the displacement are related by the equation

$$\dot{x} = \frac{P}{b} + \frac{1}{c}\dot{P}, \tag{10}$$

where b is a factor describing the viscosity of the medium and c is the factor describing elasticity.

In this case the equation of motion during impact can be obtained taking into account the relationship

$$m\ddot{x} = -P.$$

The contact force and the law of motion are determined from the equations

$$\frac{m}{c}\ddot{P} + \frac{m}{b}\dot{P} + P = 0,$$

$$\frac{m}{c}\dddot{x} + \frac{m}{b}\ddot{x} + \dot{x} = 0. \tag{11}$$

If the roots of the characteristic equation for Eq. (11) are complex numbers, the duration of impact is determined as
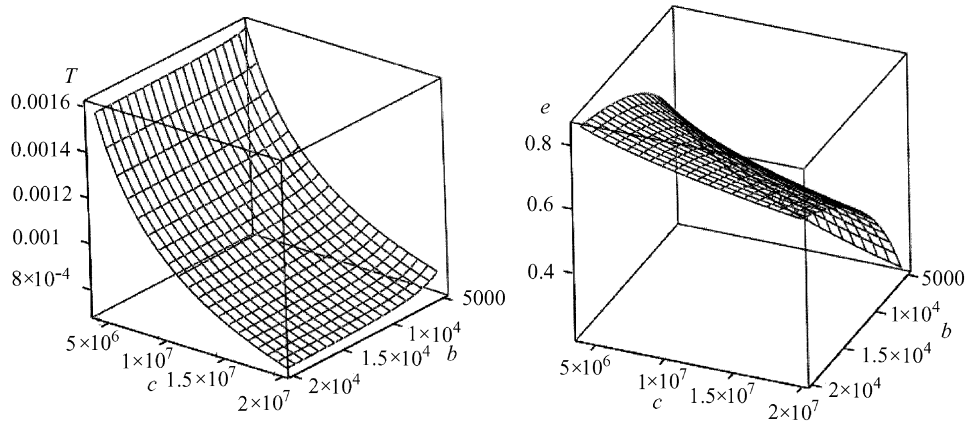
$$T = \frac{2\pi b\sqrt{cm}}{c\sqrt{4b^2 - cm}}.$$

The coefficient of restitution e does not depend on the pre-impact velocity and can be expressed in the form

$$e = \exp\left(\frac{-\pi\sqrt{cm}}{\sqrt{4b^2 - cm}}\right).$$

In Fig. 7 the plots of $T(b, c)$ and $e(b, c)$ are shown for $m = 1\,\text{kg}$.

Duration of impact is comparatively large; such a model may be useful for certain materials.

Fig. 7. Dependence of the duration of impact T and coefficient of restitution e on the coefficients b and c in case of the Maxwell medium.

5. MODELS OF IMPACT FORCES, BASED ON THE HERTZ CONTACT FORCE

Hertz investigated a static problem about impression of a rigid sphere into an elastic semi-space and showed that the elastic force was proportional to the displacement raised to the power 3/2. In accordance with this, the elastic part of the force we can take as the Hertz force and taking into account the damping force we obtain

$$m\ddot{x} = -cx^{\frac{3}{2}} - b\dot{x}. \tag{12}$$

Figure 8 shows the dependence of this force on the displacement.

The tensile force, generated in this model, can be eliminate multiplying the damping part of the force by displacement; the received equation describes the motion during impact:

$$m\ddot{x} = -cx^{\frac{3}{2}} - bx\dot{x}. \tag{13}$$

Dependence of this force and velocity on the displacement is presented in Fig. 9.

The coefficient of restitution e in the first case is 0.86 and in the second case 0.79.

We suggest the following definition of the impact force: assuming that the impact force depends both on the Hertz contact force and on the growth rate of this force, the equation of motion during impact can be written as

$$m\ddot{x} = -cx^{\frac{3}{2}} - b_1 x^{\frac{1}{2}}\dot{x}. \tag{14}$$

In this case the damping part of the force emerges as the derivative of the elastic force, multiplied by the damping factor. Damping factor b_1 represents the incorporated value, equal to $(3/2)c$ multiplied by the corresponding value of damping.

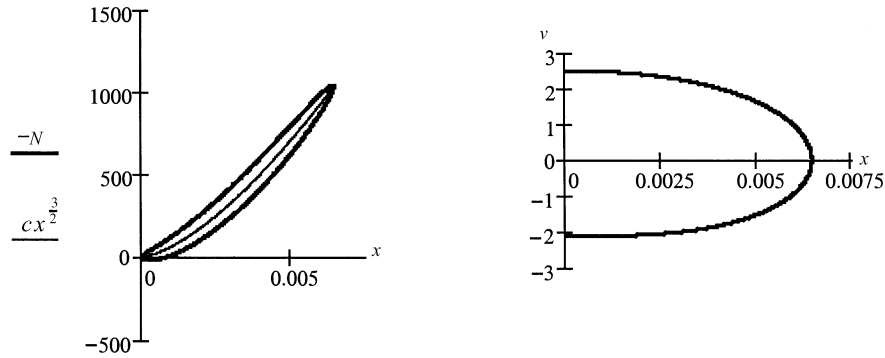In Fig. 10 the plot of the suggested impact force is presented.
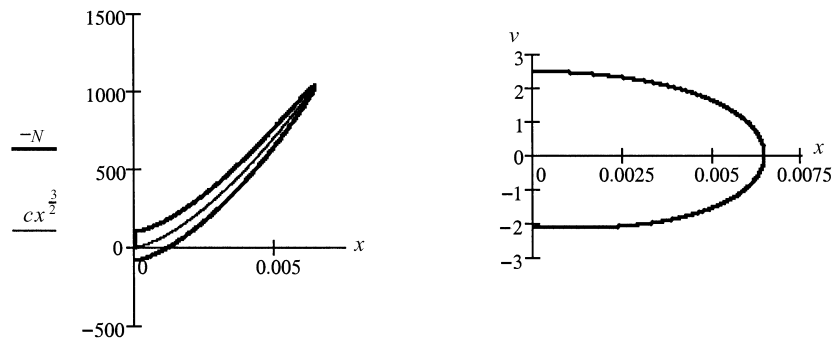
Fig. 8. Dependence of N, $cx^{3/2}$ and v on the displacement in the case of the Hertz force ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b = 40$).
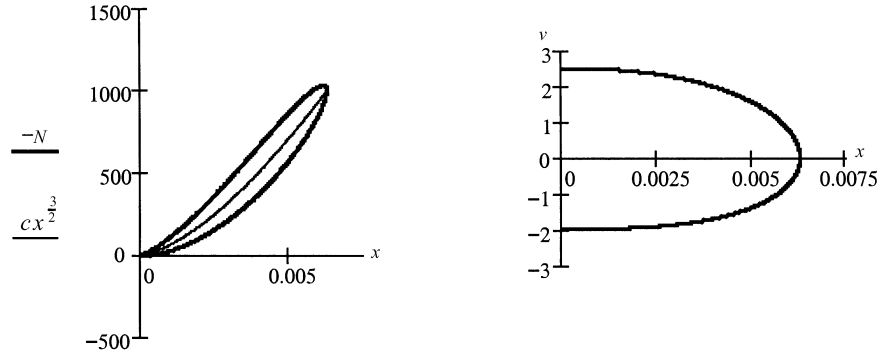
Fig. 9. Dependence of the impact force N and velocity v on the displacement in the case of the Hertz force and changed damping force ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b = 2 \times 10^4$).

Fig. 10. Dependence of the impact force N and velocity v on the displacement for the model (14) ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $b_1 = 800$); $e = 0.85$.

6. MODEL OF THE HUNT–GROSSLEY CONTACT FORCE

Hunt and Grossley [8] derived a model of the impact force with non-linear spring-damper, satisfying the boundary condition. The non-linear model is described as

$$m\ddot{x} = -k x^n - \lambda x^n \dot{x}. \tag{15}$$

The restoring part of the force is represented by the Hertz contact force and its damping part is the product of the Hertz force, the velocity of deformation and the parameter a. The equation of motion during impact is

$$m\ddot{x} = -c x^{\frac{3}{2}} - \frac{3}{2} a \dot{x} c x^{\frac{3}{2}}. \tag{16}$$

The plot of the Hunt–Grossley impact force is shown in Fig. 11.
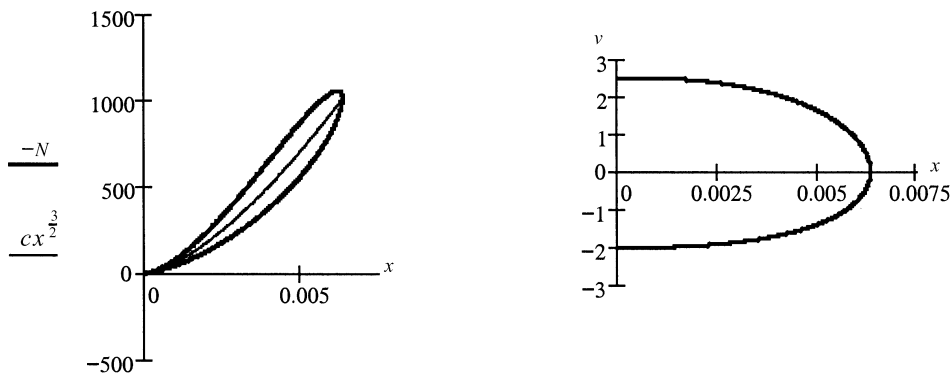
Fig. 11. Dependence of the impact force N and velocity v on the displacement in the case of the Hunt–Grossley force ($m = 1$, $c = 2 \times 10^6$, $v_0 = 2.5$, $a = 0.1$); $e = 0.80$.

7. NUMERICAL SOLUTION OF THE MOTION EQUATION OF A SYSTEM WITH IMPACTS

To determine the motion of a system with impacts one has to include the contact force in the system of equations describing the motion. It is convenient to apply models of two last forces, whose deformation curve takes the shape of a hysteresis curve. Besides the motion we shall receive also the value of the impact force. We shall consider numerical solution of the motion of a ball, falling vertically downwards under gravity with frictionless impacts against the floor and rebounding (Fig. 12).
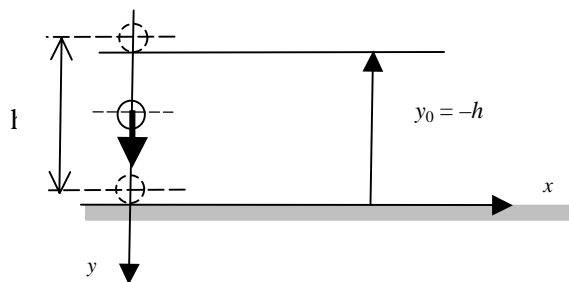
Fig. 12. Motion of a ball subject to gravity and bouncing on the floor.

The differential and algebraical equations of the motion of this body are:

$$m\ddot{x} = 0,$$

$$m\ddot{y} = mg + N,$$

$$N = 0, \quad \text{if} \quad y < 0,$$

$$N = N_{\text{el}} + N_{\text{dam}}, \quad \text{if} \quad y \geq 0, \tag{17}$$

where the impact force N consist of elastic (N_{el}) and damping (N_{dam}) parts.

There is no rotation of the body and the initial conditions are: $t = 0$, $x = 0$, $y = -h$, $\dot{x} = 0$, $\dot{y} = 0$.

The mass of the body is 1 kg, the motion begins from $h = 0.60$ m and the velocity before the first impact is 3.43 m/s.

Numerical solution of the system (17) was obtained with the Euler method, which gives acceptable results if $\Delta t = s$ is small enough; the algorithm is as follows:

$$y_{n+1} = y_n + \dot{y}_n \Delta t,$$

$$\dot{y}_{n+1} = \dot{y}_n + g\Delta t + \frac{1}{m} N_n \Delta t.$$

For the impact force in accordance with model (14) we obtain

$$N_{n+1} = (-c\, y_n^{\frac{3}{2}} - b_1 y_n^{\frac{1}{2}} \dot{y}_n) \cdot if\,(y_n \geq 0, 1, 0),$$

in accordance with model (1)

$$N_{n+1} = (-c\, y_n - b\, \dot{y}_n) \cdot if\,(y_n \geq 0, 1, 0),$$

and in accordance with model (6)

$$N_{n+1} = (-c\, y_n - b_1 y_n \dot{y}_n) \cdot if\,(y_n \geq 0, 1, 0).$$

The problem was solved with the Mathcad program. Step of integration was selected as $s = 0.00002$.

In Figs. 13 to 15 the results of numerical solution of the equations (17) with impact force in accordance with model (14), based on the Hertz force with $c = 10^6$, $b_1 = 1350$ are presented.

For comparison of the impact forces influence, the results of the numerical solution of the system (17) are given for the forces in accordance with the Kelvin–Voigt model (1) and the modified Kelvin–Voigt model (6). The solution for the impact force model (1) with $c = 2 \times 10^6$ $b = 480$ is presented in Figs. 16 to 18. The solution for the impact force model (6) with $c = 2 \times 10^6$ and $b = 8 \times 10^5$ is shown in Figs. 19 to 21.
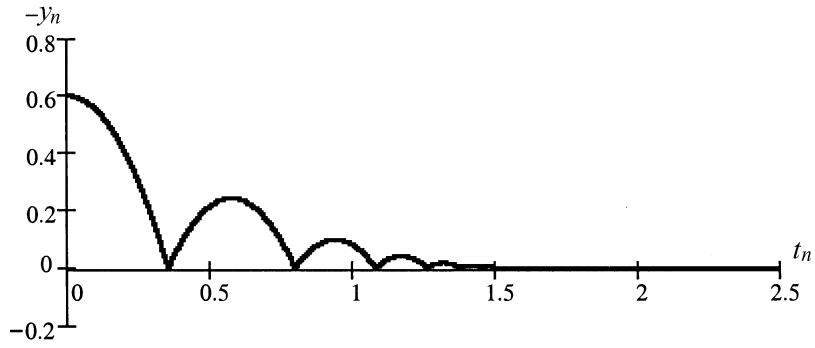
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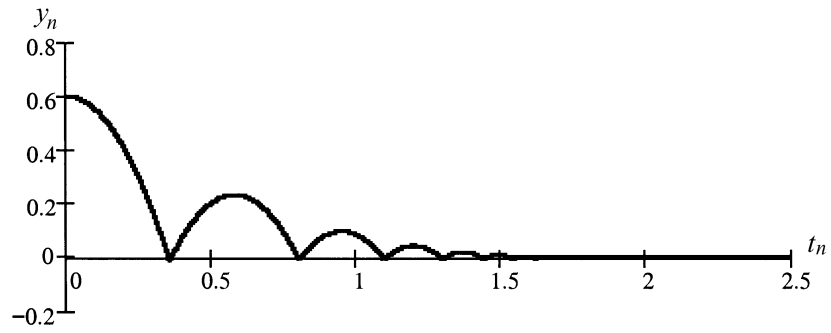
Fig. 13. Dependence of the coordinate y_n on time with impact force based on the Hertz force model.
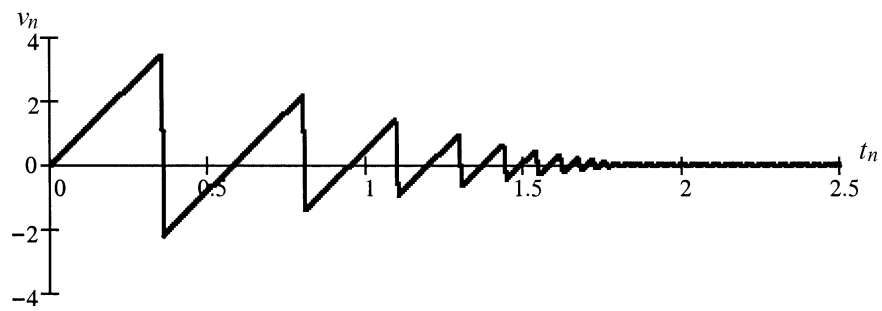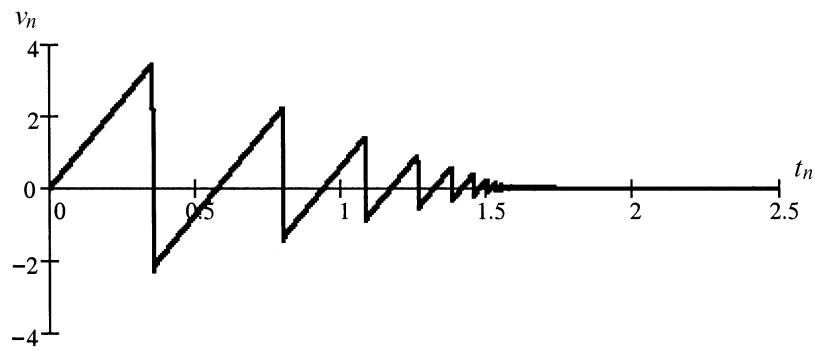

Fig. 14. Dependence of the velocity on time with impact force based on the Hertz force model.
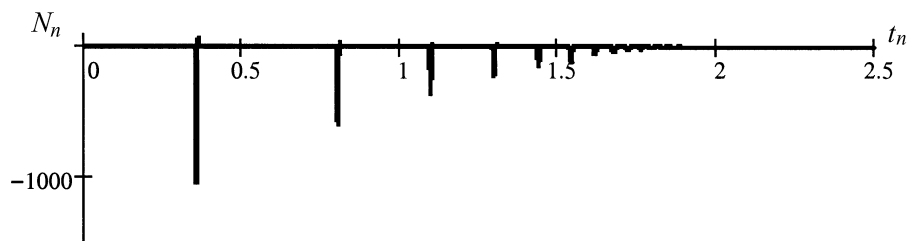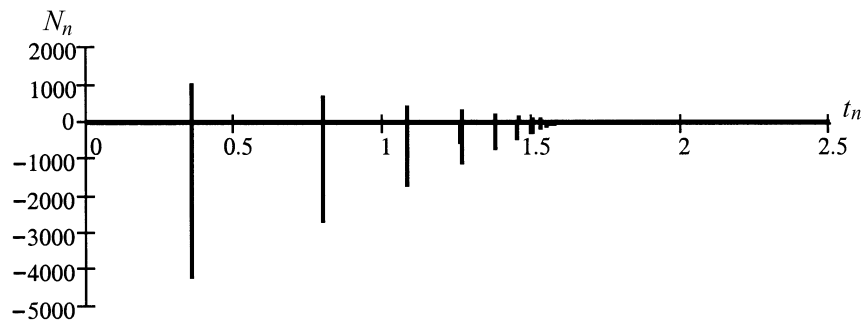

Fig. 15. Dependence of the impact force on time based on the Hertz force model.
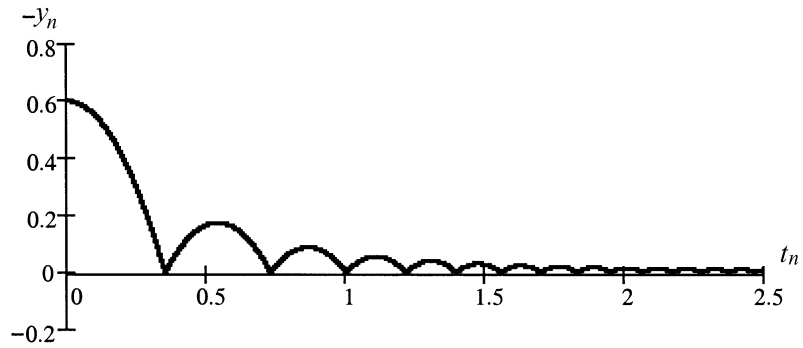
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Fig. 16. Dependence of y_n on time for the Kelvin–Voigt impact force model (1).
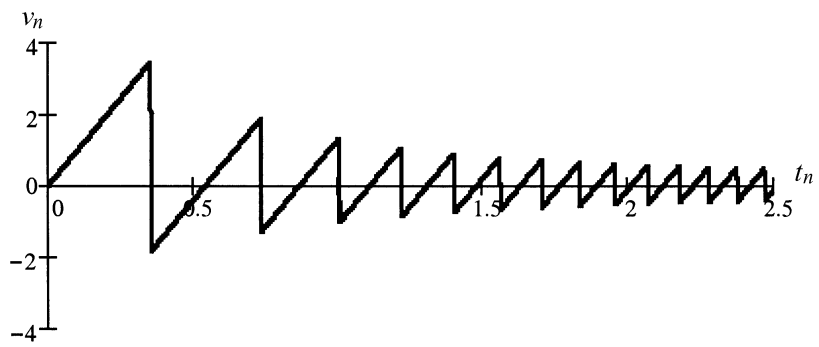

Fig. 17. Dependence of the velocity v_n on time for the Kelvin–Voigt impact force model (1).

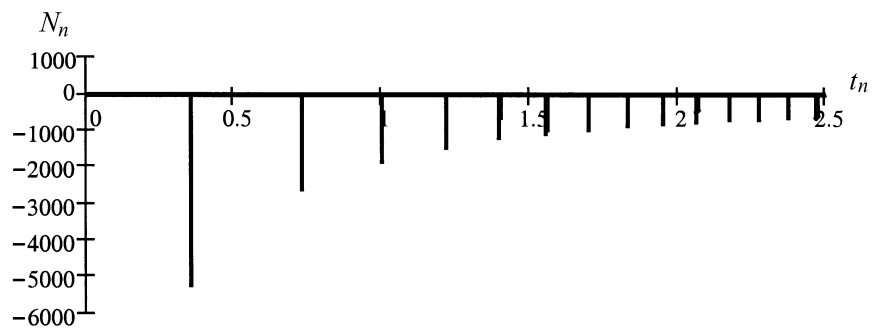**Fig. 18.** Dependence of the impact force N on time for the Kelvin–Voigt impact force model (1).

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Fig. 19. Dependence of y_n on time $y = y(t)$ for the modified Kelvin–Voigt model (6).

Fig. 20. Dependence of the velocity v_n on time for the modified Kelvin–Voigt model (6).

Fig. 21. Dependence of the impact force N on time for the modified Kelvin–Voigt model (6).

For the force models (1) and (14), the coefficients of restitution are the same $(e = 0.64)$, but impact forces are different; in both cases the impact is infinitely damped.

In the case of the model (6), the coefficient of restitution is equal to 0.54 for the first impact and for every following impact it is increased and approaches $e = 1$.

If we have experimental data about the coefficient of restitution, we can match the coefficients c and b in order to receive the proper value for e and impact time T, thus the damping and elastic parts of the impact force may be evaluated.

8. CONCLUSIONS

The models of impact forces, partially considering deformations, are the most convenient for systems with frictionless impacts, as they permit to use common differential equations of dynamics with additional contact forces for the solution of the problems of motion. The simplest Kelvin–Voigt model, which is a linear model both for stiffness and damping, shows physically impossible tensile component in the hysteresis loop.

The models of impact forces, based on the Hertzian contact theory for point contact, show a more realistic force–displacement diagram: in the unloaded initial position $x = 0$ the spring-damper force is equal to zero. This approach permits to calculate the value of impulsive forces and deformations during the impact if characteristics of the material are known. The models of suggested contact forces represent the impact cycle from the beginning of the contact to its end.

REFERENCES

1. Ivanov, A. P. *Dynamics of Systems with Mechanical Impacts*. International Program of Education, Moscow, 1997 (in Russian).
2. Glocker, Ch. On frictionless impact models in rigid-body systems. *Phil. Trans. Royal Soc. London,* A, 2001, **359**, 2358–2404.
3. Stronge, W. J. *Impact Mechanics*. Cambridge University Press, Cambridge, 2000.
4. Stoianovici, D. and Hurmuzlu, Y. A critical study of the applicability of rigid body collision theory. *J. Appl. Mech.*, 1996, **63**, 307–316.
5. Kecskeméthy, A. and Lange, C. Solving impact interactions using the regularized model approach. In *Proc. Tenth World Congress on the Theory of Machines and Mechanisms*. Oulu, 1999, **4**, 1361–1366.
6. Chatterjee, A. and Ruina, A. Two interpretations of rigidity in rigid body collisions. *J. Appl. Mech.*, 1998, **65**, 894–900.
7. Chatterjee, A. and Ruina, A. A critical study of the applicability of rigid-body collision theory. Discussion. *J. Appl. Mech.*, 1997, **64**, 247–248.
8. Kozlov, V. V. and Treschev, D. V. *Billiards. Genetic Introduction to Dynamics of Systems with Impacts.* Moscow University Press, Moscow, 1991 (in Russian).

Jäikade kehade põrke mudelid, mis osaliselt arvestavad deformatsioone

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Põrke mudelid, mis arvestavad kehade deformeerumise iseloomu, on kõige sobivamad hõõrdevaba põrke kirjeldamiseks, kuna need võimaldavad kasutada harilikke diferentsiaalvõrrandeid. Lihtsaim lineaarne Kelvini–Voigti mudel, mis arvestab küll nii jäikust kui ka dissipatsiooni, annab hüstereesi silmuse läbimise tulemusena füüsikaliselt ebareaalse tõmbejõu. Põrke mudelid, mis põhinevad punktkontakti korral tekkivatel Hertzi kontaktjõududel, annavad palju realistlikuma diagrammi jõud–siire, sest koormusvabas algasendis on jõud vedrus/amortisaatoris võrdne nulliga. Juhul kui materjali karakteristikud on teada, võimaldavad esitatud mudelid leida impulssjõu ja deformatsiooni väärtused põrke suvalisel hetkel. Esitatud kontaktjõudude mudelid võimaldavad kirjeldada põrketsükli kulgu kogu tsükli vältel.